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ELEMENTARY THEORY AND CALCULATION OF IRON BRIDGES AND ROOFS.

BY

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TRANSLATED FROM THE GERMAN (THIRD EDITION)

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TRANSLATOR'S PREFACE.

THE first edition of Professor Ritter's work, of which the following is a literal translation, was published in 1862 * to advocate the use of the "Method of Moments" in calculating the stresses in bridges and roofs. This "Method of Moments" is in reality but an application of Rankine's "Method of Sections." The adaptation of the method to various cases is explained and illustrated by means of numerical examples comprising several of the forms of bridges and roofs in general use as well as others not often met with.

Some interesting problems are discussed in the Eleventh Chapter, and are possibly not generally known in this country. It is required to determine the form a structure should have to fulfil given conditions as regards the stresses. These problems give a considerable insight into the manner in which the stresses are distributed amongst the various bars of a structure, and show also that comparatively small changes in the form may produce great changes in the stresses. The effect of changes of temperature on the deflection and on the stresses in a "composite structure" is treated at some length in the Fourteenth and Sixteenth Chapters. The theory of loaded beams is only touched upon—in fact, only those cases are considered which are required in the various examples.

The Sixteenth Chapter contains a very instructive example of a composite structure consisting of a pair of braced girders combined with suspension chains. It should be observed, however, that Herr Hugo B. Buschmann, in a pamphlet 'On

* The substance of the first two chapters was published previously in the 'Zeitschrift des Architecten- und Ingenieur-Vereins für das Königreich Hannover,' vol. vii., No. 4.

the Theory of Combined Girder and Suspension Bridges,' takes exception to the manner in which the equations (§§ 56 to 61) giving the stresses in the girders produced by the moving load are arrived at. In the preface to the third edition Professor Ritter says: "Herr Buschmann maintains that these equations depend on arbitrary assumptions, and thinks to prove their unsoundness by remarking that under certain conditions of loading, namely, when both ends of the girders are loaded, they give a negative bending moment at the centre of the girders, or, in other words, the girders would be bent upwards. Thus the radii of curvature for the central part of the suspension chains would be increased, and this requires a *diminution* of the length of these chains, which is evidently absurd. This conclusion is, however, incorrect. Herr Buschmann overlooks the fact that exceedingly small changes in the form of the suspension chains are under consideration, and that therefore not only the vertical but also the horizontal displacement of each element of the chain must be taken into account. Without doubt if the chord of the arc whose radii of curvature are increased did not alter, the length of this arc would be diminished. But if at the same time the length of the chord increases, not only may the length of the arc remain unaltered, but it may even be lengthened; and this is what in reality occurs, owing to the horizontal displacement of each point of the chains, not only in the case under consideration, but also for all positions of the loads."

In some few instances Professor Ritter does not agree with the more recent English practice, notably so in his estimate of the wind-pressure on roofs. These instances have been pointed out in notes added to the text and in an Appendix.

H. R. S.

Gibraltar, 1879.

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CALCULATION OF THE STRESSES IN BRIDGES AND ROOFS.

FIRST CHAPTER.

PRELIMINARY REMARKS.

roofs the design should be such that the
nployed is the smallest possible, not only
the cost of materials is great in com-
ur, but more especially because the dead
ssarily increased, and the very success of
possibly depend on the smallness of this

i structure, the maximum safe resistance
l be called forth in every part, and no-
e any unnecessary excess of material.
d thus: When the structure is placed in
s to loading, the intensity of stress in
equal to what is considered the safe re-
to which it is subject when under these
co.

This can d in the case of bars which are under direct tension or compression, for then the stress is uniformly distributed over the whole sectional area; but in the case of a beam under bending stress, it cannot be complied with, because the stresses are not uniformly distributed over the cross-section.

Therefore, in a good construction, the various parts should be, if possible, either in direct tension or compression, and bending stress should be avoided.

These views are more or less carried out in practice, and the larger the structure, the nearer is the approximation. The

endeavour to save material has led from the massive beams of rectangular section to those of I and II section, and when further the solid web was replaced by braces, those combinations of bars were arrived at in which only direct tension or compression exist. The iron roofs and the braced girders of modern times are examples of such structures.

To comply rigidly with the above conditions, the joints should be made with single bolts (pin joints). If a bar be connected to another by a single bolt, it can turn freely about its end, but if the joint be made with rivets, the end of the bar is fixed, and will therefore be subject to a slight amount of bending stress. Thus, with rivetted joints, the material is not employed to the best account; and, especially in the case of large, important structures, there is the disadvantage that the maximum stresses are not accurately known, whilst if the structure were theoretically correct, these stresses could be ascertained to the greatest degree of accuracy. It is worth noticing that the theoretical structures are also the easiest to calculate.

In all the following examples it will be assumed that the joints are hinged, the connections being made by single bolts. It will also be supposed that these joints are the only points of loading. This distribution of the load can always be obtained in practice by using bearers to bridge over the space between the joints. Whether it is advisable to construct these bearers as separate parts or to fuse them into the main structure, is a question that will be considered further on.

As regards the weight of the structure itself, it will be considered as evenly distributed over the span, and in accordance with the above, concentrated at the joints; the degree of accuracy of this assumption will be tested in the sequel.

[NOTE.—There is no doubt but that hinged connections made by means of single pins would be theoretically more perfect than rivetted joints, if a perfectly uniform distribution of the stress were the only consideration. But it is found that in structures subject to vibrations, the pins in many cases shake loose, and the holes in the bars become elliptical, owing to the hammering action that takes place between the pins and the faces of the holes, and this action will always occur unless the pins are made a *drawing fit* in the holes. This is the case, for instance, in the central joints of a railway bridge, where (as will be seen) the stresses are constantly changing from tension to com-

pression, and *vice versâ*. It may be mentioned that this action occurred in the Crumlin Viaduct, and that in consequence gusset plates had to be added. Pin joints may however often be used with advantage, both with regard to economy, simplicity of erection, and appearance in structures subject to a purely dead load, or even to a live load, if unaccompanied by vibrations and rapid changes in the nature of the stresses, as, for instance, in the case of roofs.

The objections raised to rivetted joints by Professor Ritter apply in reality only to those as usually designed, for the arrangement of the rivets in a joint can be such that little or no bending stress occurs in the bars connected. This was pointed out by Professor Callcott Reilly in two papers read before the Institution of Civil Engineers.* Premising the following definition—"The mean fibre of a bar is the line passing through the centres of gravity of all cross-sections, and is consequently one of the axes of gravity of the bar," the rules according to which rivetted joints should be designed are thus stated by Professor Reilly:—

1st. The mean fibres of any two or more members of a truss connected by a group of rivets must intersect at one point.

2nd. The group of rivets connecting the bars must be arranged symmetrically round this point of intersection of the mean fibres of the bars; or in other words, the resultant resisting force of the group of rivets must occur at the intersection of the mean fibres of the bars connected by the said group.

3rd. The first row of rivets in each bar, that is, the row on the side towards which the stress is transmitted, must be symmetrical with the mean fibre of that bar.

If the stress is uniformly distributed over the cross-section of any bar, the resultant stress must lie in the mean fibre; it is therefore evident that unless the mean fibres of the bars connected intersect in a point, the stress, in some of them at least, will not be uniformly distributed.

The resultant pull or thrust of a bar must evidently lie in the same straight line as the resultant resistance of the rivets connecting the bar. If, therefore, rule 2 be not complied with, the resultant pull or thrust will not pass through the mean fibre, and evidently the stresses will not be uniformly distributed but will be uniformly varying, and therefore more intense on one edge (the edge nearest the resultant) than upon the other. In a similar manner the stress will not be uniformly distributed if the first row of rivets be not symmetrical with the mean fibre.

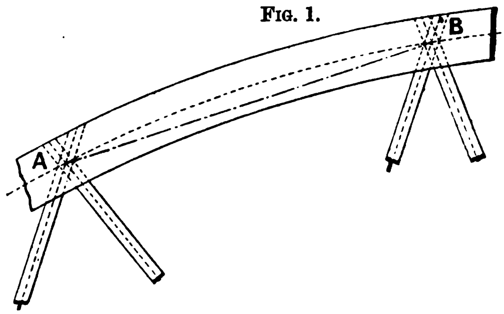
To obtain a theoretically perfect joint, every row of rivets should be symmetrical with the mean fibre. Such an arrangement can, however, only be obtained when two bars cross at right angles. But the first row of rivets, for instance, relieves the part of the bar beyond of a certain amount of stress; therefore unless the second row of rivets be very much displaced, the greatest intensity of stress at the section through this row will not reach the intensity or stress in the bar before the leading rivets. This is evidently, *a fortiori*, true of the 3rd, 4th, &c., rows of rivets. The rules given above are therefore sufficient for practical purposes.

* 'Minutes of Proceedings,' vols. xxiv. and xxix.

Rivetted joints have also this advantage over pin joints, that the ends of the bars connected can be considered "fixed," and this materially increases the resistance of those bars subject to compression. Pin joints are also as a rule more expensive than rivetted joints, but easier to put together by unskilled labour.

It will be observed that the mean fibre need not necessarily be a straight line; but if it is curved no arrangement of the joints will make the stress uniformly distributed at every cross-section. This is, for instance, the case if the bow of a bowstring girder is curved between the joints. Fig. 1 represents

FIG. 1.

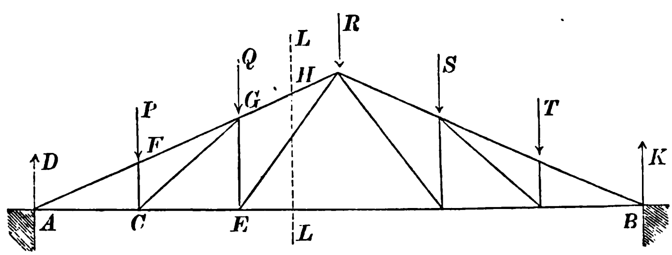


a portion of the top boom of such a girder, and the mean fibres of the various bars are indicated by dotted lines. The thrust in the booms must evidently act in the straight line joining A and B, and must therefore give rise to bending stress, or, in other words, the stress will not be uniformly distributed.]

§ 2.—METHOD OF MOMENTS.

The method adopted to calculate the stresses in the various structures given in the following examples is known as the

FIG. 1 (a).



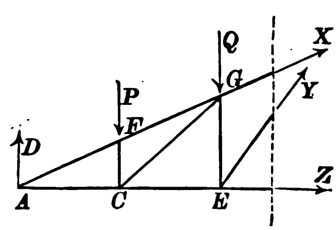
"method of moments," and it will be explained by means of the roof represented by Fig. 1 (a).

The total load on this roof consists of the five single loads, P, Q, R, S, T, which are to be considered as weights hung to the top joints. These five loads produce the reactions D and K at the points of support; the sum of these reactions must be equal to the whole load, and they can easily be determined by the ordinary rules of statics. If, for instance, the span A B is divided by the verticals through the weights into six equal parts,

$$D = \tfrac{1}{6}T + \tfrac{2}{6}S + \tfrac{3}{6}R + \tfrac{4}{6}Q + \tfrac{5}{6}P.$$
$$K = \tfrac{5}{6}T + \tfrac{4}{6}S + \tfrac{3}{6}R + \tfrac{2}{6}Q + \tfrac{1}{6}P.$$

Imagine the combination of bars divided into two parts by the line L L, then each part (Fig. 2, for instance) can only be retained in equilibrium by applying to each bar at the point of section a force which represents the action of the other part. *This force must lie in the direction of the bar*, for otherwise the bar would rotate round its end; this force is, in fact, what is called the stress in the bar. Thus the stresses X, Y, Z in the three bars, which have been cut through, together with the remaining loads D, P, Q, are in equilibrium. All these forces lie in the same vertical plane, and they therefore must satisfy the three following conditions of equilibrium:—

Fig. 2.



1. The sum of the vertical forces acting upwards must be equal to the sum of the vertical forces acting downwards.

2. The sum of the horizontal forces acting towards the right must be equal to the sum of the horizontal forces acting towards the left.

3. The sum of the statical moments of all the forces tending to turn the part of the roof represented by Fig. 2 round any point from right to left, must be equal to the sum of the statical moments of those forces tending to turn it from left to right round the same point; for the part of the roof under consideration can be regarded as a lever, and any point can be taken as the fulcrum.

These three conditions can be expressed more concisely by the equations

$$\Sigma (H) = 0, \quad \Sigma (V) = 0, \quad \Sigma (M) = 0,$$

where H and V are the resolved parts of any force horizontally and vertically respectively, M the moment of a force round any point, and Σ denotes that the forces or moments have been added together algebraically, that is, the sign of each force or moment is taken as plus or minus according to the direction in which it acts. The three stresses X, Y, and Z will be contained in each of these equations, and by solving them the values of X, Y, and Z can be obtained. The stresses in all the members of the roof can be similarly ascertained by taking other sections.

This method can always be applied, but it has two serious defects. The first is, that H and V contain the cosine and sine of the angles the bars make with the horizontal, and these angles must therefore be determined. The second is, and it is more serious than the other, that in order to ascertain any one stress all three equations have, as a rule, to be solved.

There is, however, a very simple method, which can be applied to all cases, and which is free from the above defects. Apart from this, the method has the advantage of requiring only the application of the principle of the lever (in its more general form the law of statical moments), and can therefore be easily understood by those who are acquainted but with the very elements of mechanics. In fact only the last equation, that of statical moments, need be used, for if to obtain the stress in one bar moments are taken round the point of intersection of the other two bars, an equation will be arrived at containing only one unknown, the stress required, for evidently the moments of the stresses in the other two bars vanish.

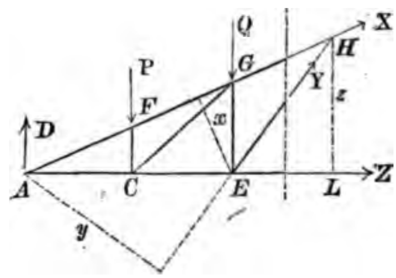
The lever arms of the various forces will have to be determined, and this can be done with sufficient accuracy from a drawing to scale.

A general rule, framed from the above, can be thus stated:—

Consider the structure divided into two parts by a section,

which if possible should only cut through three bars, and apply the forces X, Y, Z to these bars to maintain equilibrium, these forces being the stresses in the bars. Form the equation of moments for either part of the structure, and if X is to be determined the point of intersection of Y and Z is to be chosen as the point round which to take moments, if Y the point of intersection of X and Z, and if Z the point of intersection of X and Y.

Fig. 3.



For instance, in the above example to determine X, the point round which to take moments would be E, the intersection of Y and Z (Fig 3). The equation of moments is

$$X x - P.CE + D.AE = 0,$$

or

$$X = \frac{P.CE - D.AE}{x}.$$

To determine Y take moments round A, the point of intersection of X and Z, thus:

$$- Yy + P.AC + Q.AE = 0,$$

or

$$Y = \frac{P.AC + Q.AE}{y}.$$

And to determine Z take moments round H, the point of intersection of X and Y, thus:

$$- Z z - Q.EL - P.CL + D.AL = 0,$$

or

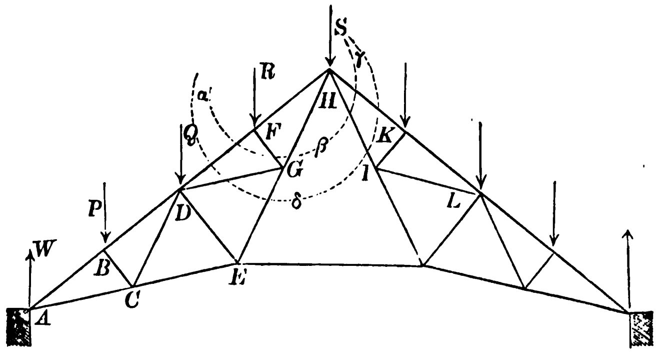
$$Z = \frac{- Q.EL - P.CL + D.AL}{z}.$$

This method can be directly applied to all structures in

which it is possible to reach each bar by a section that does not cut through more than three bars.

In some cases, however, in the truss shown in Fig. 4, for instance, there may be bars which cannot be reached by sections cutting only through three bars; such are the bars F G, D G, D E.

FIG. 4.



But even in such a case the method may be *directly* applied if all the bars cut through by the section (which may be curved or straight) intersect in a point except the one the stress in which is to be determined.

For instance, to find the stress V in the bar F G, take a

FIG. 5.

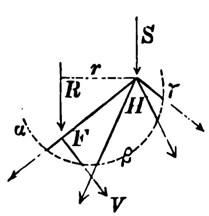
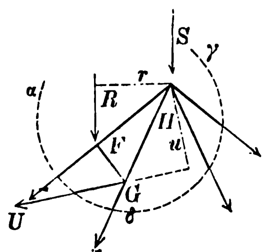


FIG. 6.



section $\alpha\,\beta\,\gamma$ and form the equation of moments about the point H for the part cut out (Fig. 5).

$$-\,\mathrm{V}\,.\,\mathrm{FH} - \mathrm{R}\,r = 0,$$

or

$$\mathrm{V} = -\,\frac{\mathrm{R}\,r}{\mathrm{F\,H}}.$$

In the same manner the stress U in the bar D G can be ascertained by taking a section $\alpha\,\delta\,\gamma$ and forming the equation of moments round the point H for the part cut out, thus:

$$U u - R r = 0,$$

or,

$$U = \frac{R r}{u}.$$

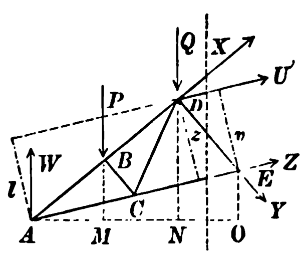
Similarly the stresses in K J and L J can be found. The remaining bars can all be reached either by sections which only cut through three bars, or else by sections which cut through four bars but the stress in one of which is already known. In both cases the method of moments can be applied.

Thus, when the stress U is known the stresses X, Y, Z in the bars D F, D E, C E, can be found from the equations

$$\begin{aligned} &X . DE + U \check{r} - Q . NO - P . MO + W . AO = 0, \\ &Y . AD + U l + Q . AN + P . AM = 0, \\ &- Z z + W . AN - P . MN = 0, \end{aligned}$$

obtained by taking moments round the points, E, A, and D respectively.

FIG. 7.



This more complicated example shows the advantages of the proposed method. They become even more apparent when it is considered that only the beginner will require to make separate figures for each calculation. The adept will easily form the equations from the principal drawing.

The general method having now been explained, its application to various cases will be best seen by means of numerical examples. It will be sufficient to give the complete calculations for a few bars only, those which can be considered as the representatives of others similarly situated. For the remaining bars only the equation of moments and the results will be given.

It is of no consequence which direction of rotation is taken as the positive one, but to avoid errors some direction should be chosen; it will be considered in the sequel that rotation from

left to right is positive,* and that rotation from right to left is negative.

Further, all stresses will be considered as pulling stresses (this has already been done in the former examples), therefore *positive stresses will represent tension* and *negative stresses compression.*

This, it will be observed, is the reverse to the usual English practice.

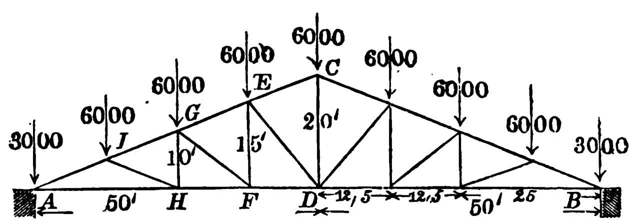
[NOTE.—A great deal of clerical labour can be spared by rightly choosing the scale by which the lever arms are to be measured. This remark refers principally to structures divided into bays of equal length. It will probably be best in this case to make the length of each bay unity, when it will be found that the lever arms of the various loads are generally whole numbers. This plan it will be seen has been adopted in several of the examples given. If the loads on the structure are placed at equal intervals, the horizontal distance between them should be taken as unity.]

§ 3.—CALCULATION OF THE STRESSES IN A ROOF OF 100 FEET † SPAN.

Drill-shed of the Welfenplatz Barracks, Hanover.

The weight of the roof covering and framing (Fig. 8) is 11·3 lbs.† per square foot of horizontal surface covered, and

FIG. 8.



20 lbs. more per square foot must be added for snow and wind pressure.‡ The total load is therefore 31·3 lbs. per square foot of horizontal surface.

The distance apart of the principals is $15\frac{1}{3}$ feet, and since the span is 100 feet, $15\frac{1}{3} \times 100$ square feet of horizontal surface is supported by each principal, and the load on each is $15\frac{1}{3} \times 100 \times 31 \cdot 3$ lbs., or in round numbers 48,000 lbs. The

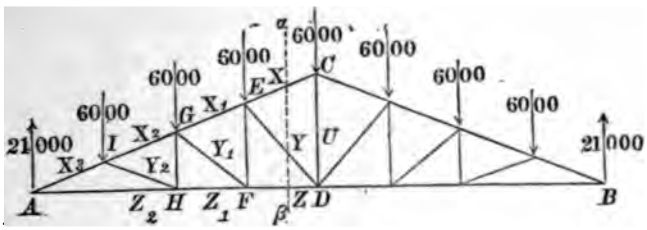
* In the same direction as the hands of a watch.

† German feet and lbs.

‡ This estimate of the snow and wind pressure does not agree with the latest English practice. See Appendix.

load on each of the eight divisions of the roof is therefore 6000 lbs. It may be considered that one-half of the 6000 lbs. on each division is applied at each of the two adjacent joints, and this can be effected by means of bearers or common rafters.* The load on the seven central joints will therefore be 6000 lbs., and on each of the end joints 3000 lbs. Evidently the load on the end joints will be taken up directly by the abutments. The reaction at each abutment is altogether 24,000 lbs., and subtracting the 3000 lbs. on the end joint, the pressure against the combination of bars is 21,000 lbs.

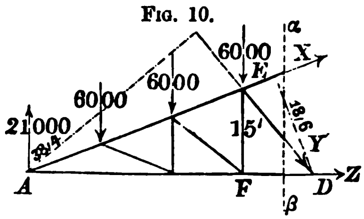
FIG. 9.



The structure is therefore subject to the action of nine exterior forces; seven of 6000 lbs. each acting downwards on the central joints, and two of 21,000 lbs. each acting upwards on the end joints.

To find the stresses X, Y, Z, in the bars of the central bay

FIG. 10.



(Fig. 9) let the roof be divided into two parts by a section $\alpha\ \beta$

* This distribution of the load requires the common rafters to be articulated at each joint. They are, however, generally continuous, and this slightly alters the distribution of the load, for then part of the load is transmitted directly to the abutments by the common rafters. It is, however, usual in practice to adopt the above distribution, the error being on the side of safety. See 'Lectures on the Elements of Applied Mechanics,' by Morgan W. Crofton, F.R.S., and 'Instruction in Construction,' by Col. Wray, R.E.—TRANS.

and the forces X, Y, Z applied to maintain equilibrium. To obtain X, consider the part shown in Fig. 10 as a lever with its fulcrum at D, the point of intersection of Y and Z; then for equilibrium the following equation of moments must hold: *

$$0 = X \times 18{\cdot}6 + 21{,}000 \times 50 - 6000 \times 12{\cdot}5 - 6000 \times 25 - 6000 \times 37{\cdot}5,$$

whence

$$X = -32{,}300 \text{ lbs.}$$

Similarly to find Y take moments round A, the point of intersection of X and Z, thus:

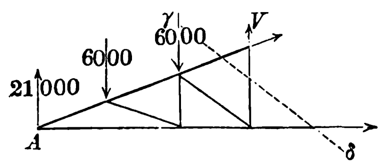
$$0 = Y \times 38{\cdot}4 + 6000 \times 12{\cdot}5 + 6000 \times 25 + 6000 \times 37{\cdot}5$$
$$Y = -32{,}300 \text{ lbs.}$$

And to obtain Z take moments round the point E:

$$0 = -Z \times 15 + 21{,}000 \times 37{\cdot}5 - 6000 \times 12{\cdot}5 - 6000 \times 25$$
$$Z = +37{,}500 \text{ lbs.}$$

To find the stress in the vertical rod E F take an oblique

FIG. 11.



section $\gamma\ \delta$ (Fig. 11), and the equation of moments round A, the point of intersection of the other two bars intersected by $\gamma\ \delta$ will be

$$0 = -V \times 37{\cdot}5 + 6000 \times 12{\cdot}5 + 6000 \times 25,$$

whence

$$V = +6000 \text{ lbs.}$$

The equations for the similarly situated bars can be formed in like manner, thus:—

$$0 = X_1 \times 13{\cdot}9 + 12{,}000 \times 37{\cdot}5 - 6000 \times 12{\cdot}5 - 6000 \times 25.$$
$$X_1 = -40{,}400 \text{ lbs. (†Turning point F).}$$
$$0 = Y_1 \times 23{\cdot}5 + 6000 \times 12{\cdot}5 + 6000 \times 25.$$
$$Y_1 = -9570 \text{ lbs. (Turning point A).}$$
$$0 = -Z_1 \times 10 + 21{,}000 \times 25 - 6000 \times 12{\cdot}5.$$
$$Z_1 = +45{,}000 \text{ lbs. (Turning point C).}$$

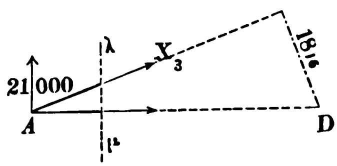
* The method of finding the lever arms by calculation is given in the eleventh section of this book.

† The "turning point" (Drehungspunkt) is the point with reference to which the equation of moments is formed.

$$0 = -V_1 \times 25 + 6000 \times 12 \cdot 5 \text{ (Turning point A).}$$
$$V_1 = +3000 \text{ lbs.}$$
$$0 = X_2 \times 9 \cdot 3 + 21{,}000 \times 25 - 6000 \times 12 \cdot 5.$$
$$X_2 = -48{,}400 \text{ lbs. (Turning point H).}$$
$$0 = Y_2 \times 9 \cdot 3 + 6000 + 12 \cdot 5 \text{ (Turning point A).}$$
$$Y_2 = -8100 \text{ lbs.}$$
$$0 = Z_2 \times 5 + 21{,}000 \times 12 \cdot 5 \text{ (Turning point J).}$$
$$Z_2 = +52{,}500 \text{ lbs.}$$

The stress in X_3 is to be found by means of the section $\lambda\,\mu$ (Fig. 12), which only intersects two bars. In such a case any

FIG. 12.



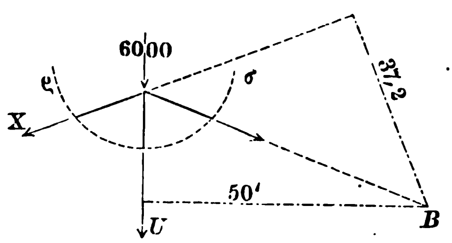
point in the other bar can be chosen as turning point.

Thus taking moments round D,

$$0 = X_3 \times 18 \cdot 6 + 21{,}000 \times 50.$$
$$X_3 = -56{,}500 \text{ lbs.}$$

The central vertical bar is the only one the stress in which cannot be found directly. To obtain this stress, that in one of

FIG. 13.



the adjacent bars must be known. Thus it has been found that X = − 32,300 lbs.; hence (Fig. 13) the equation of moments about B to find the stress U is

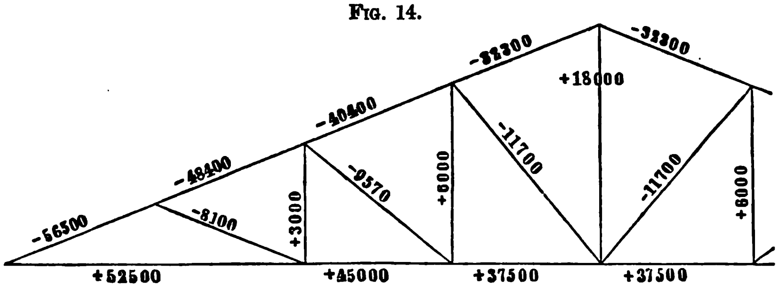
$$0 = -U \times 50 - 6000 \times 50 - (-32{,}300) \times 37 \cdot 2,$$

whence

$$U = +18{,}000 \text{ lbs.}$$

The results of the above calculations are given in Fig. 14.

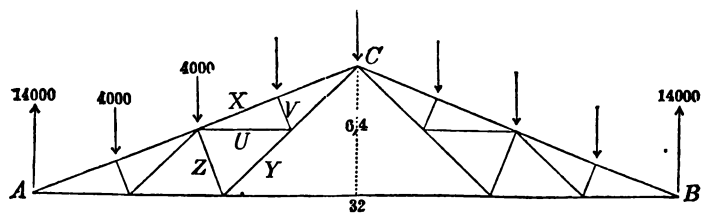
FIG. 14.



§ 4.—ROOF TRUSS OF 32 METRES SPAN.

The total load on the roof truss represented in Fig. 15 is assumed to be 32,000 kilos., or 1000 kilos. per metre of span. Proceeding as in the previous example, it is found that there is on each central joint a load of 4000 kilos. and an upward pressure of 14,000 kilos. at each abutment.

FIG. 15.



It will be seen that to find the stresses in the bars V, U, X, Y, and Z, the variation of the method of moments explained at the end of § 2 must be employed.

For instance, to calculate the stress V the portion of the roof shown in Fig. 16 must be considered, and taking moments round C the equation

$$0 = -\mathrm{V} \times 4{\cdot}308 - 4000 \times 4$$

is obtained, whence

$$\mathrm{V} = -3714 \text{ kilos.}$$

Similarly the stress U can be found from the portion of the roof shown in Fig 17, thus

$$0 = U \times 3{\cdot}2 - 4000 \times 4, \quad \text{or} \quad U = +\ 5000 \text{ kilos.}$$

Having determined U, the stress X can be found from Fig. 18 by taking moments round E:

$$0 = X \times 3{\cdot}4465 + 14000 \times 9{\cdot}28 - 4000\,(1{\cdot}28 + 5{\cdot}28) + 5000 \times 3{\cdot}2,$$
$$\text{or} \quad X = -\ 34{,}725 \text{ kilos.}$$

FIG. 16.

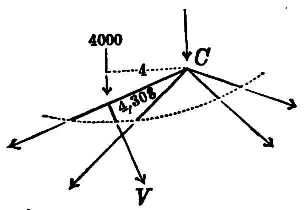
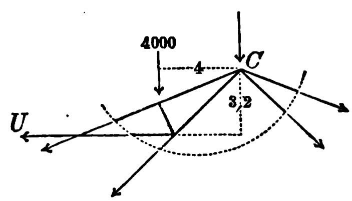


FIG. 17.



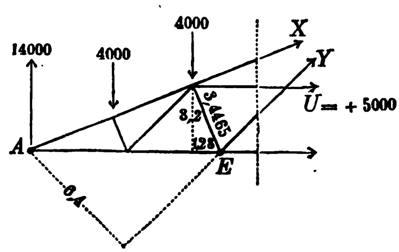
Likewise Y can be found by taking moments round A:

$$0 = -\ Y \times 6{\cdot}4 + 4000\,(4 + 8) + 5000 \times 3{\cdot}2,$$
$$\text{or} \quad Y = 10{,}000 \text{ kilos.}$$

To determine the stress Z (Fig. 15), an oblique section passing to the left of the point E (Fig. 18) must be drawn, and by taking moments round A,

$$0 = Z \times 8{\cdot}616 + 4000\,(4 + 8) + 5000 \times 3{\cdot}2,$$
$$\text{or} \quad Z = -\ 7428 \text{ kilos.}$$

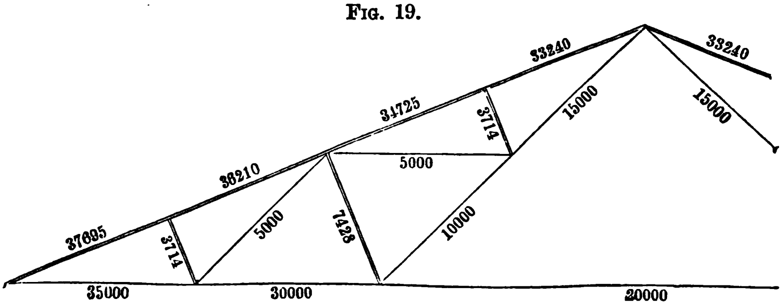
FIG. 18.



The remaining nine bars of the left half of the roof can each be reached by a section intersecting only three bars, and the

stresses in them can therefore be calculated in the manner shown in the previous numerical example. The results of these cal-

Fig. 19.



culations are given in Fig. 19, and as the bars in compression have been drawn with double lines, the signs have been omitted.

SECOND CHAPTER.

§ 5.—Application of the Method of Moments to the Calculation of Bridges.

One great advantage of the method described in the previous pages is that the stress in any particular bar can be found at once by means of a single equation. But there is yet another advantage which adapts this method more particularly to the calculation of Bridges. It is this: that from the inspection of *one* equation of moments it is possible to ascertain what loads on the bridge increase the stress in any particular bar and what loads decrease it. Therefore to find the maximum stress in a bar it is only necessary to leave out of the equation those loads which diminish the stress. And to find the minimum stress (which in some cases will be compression) those loads which increase the stress must be omitted. It is unnecessary to add that the above has reference to temporary loads only.

This does not apply to the previous examples, for—as can be easily ascertained—the removal of any of the loads does not increase the stress (either tension or compression) in any of the bars. In the case, however, of the structures that are usually adopted for bridges and also in some roof trusses (as will appear further on), it is of great importance to ascertain the effect of the variation of the loading,* for the greatest stress (either of tension or compression) may not occur when the structure is fully loaded.

[Throughout, the term greatest stress is used irrespective of the sign of the stress, but the terms maximum and minimum depend on the sign, thus the minimum stress may be the greatest compression.]

* For example, the temporary load produced on a bridge by a train, or in the case of roofs, by the snow or wind-pressure, applied to one side only.

In the girder shown in Fig. 20, for instance, the stress S is found by taking a section M N and forming the equation of moments round O

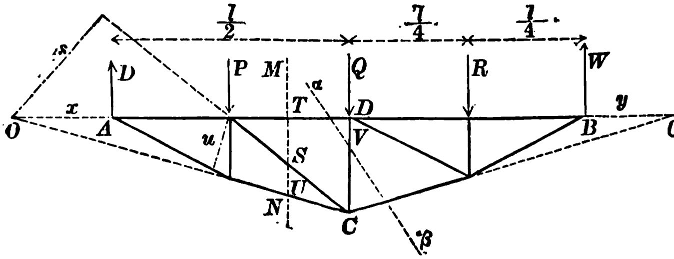
$$\mathrm{S}s - \mathrm{D}x + \mathrm{P}\left(x + \frac{l}{4}\right) = 0;$$

or substituting for D its value: $\frac{3}{4}\mathrm{P} + \frac{\mathrm{Q}}{2} + \frac{\mathrm{R}}{4}$;

$$\mathrm{S} = \frac{-\mathrm{P}\left(\frac{x}{4} + \frac{l}{4}\right) + \frac{\mathrm{Q}}{2}x + \frac{\mathrm{R}}{2}x}{s}.$$ 4

The member containing P is negative, and the members containing Q and R are positive. Evidently then the load P

FIG. 20.



diminishes the stress and the loads Q and R increase it.

Hence the equation

$$\mathrm{S}\,(\text{max.}) = \frac{\frac{\mathrm{Q}}{2}x + \frac{\mathrm{R}}{4}x}{s}$$

gives the greatest tension produced, and the equation

$$\mathrm{S}\,(\text{min.}) = \frac{-\mathrm{P}\left(\frac{x}{4} + \frac{l}{4}\right)}{s}$$

gives the greatest compression.

For simplicity it has been considered that P, Q, R are moving loads, and that the girder itself has no weight.

The equations for T and U are

$$T \, . \, CD - P\frac{l}{4} + D\frac{l}{2} = 0;$$

$$- U u + D\frac{l}{4} = 0;$$

or substituting for D its value and solving

$$T = \frac{-P\frac{l}{8} - Q\frac{l}{4} - R\frac{l}{8}}{CD};$$

$$U = -\frac{P\frac{3l}{16} + Q\frac{l}{8} + R\frac{l}{16}}{u};$$

from which it appears that the greatest stresses in these bars occur when the girder is fully loaded.

The equation to find the stress V is (Section $\alpha\,\beta$. Turning-point O,)

$$- V\left(y + \frac{l}{2}\right) - Q\left(y + \frac{l}{2}\right) - R\left(y + \frac{l}{4}\right) + W y = 0,$$

or substituting for W its value: $\frac{3}{4}R + \frac{Q}{2} + \frac{P}{4}$

$$V = \frac{-R\left(\frac{y+l}{4}\right) - Q\left(\frac{y+l}{2}\right) + P\frac{y}{4}}{y + \frac{l}{2}},$$

whence as before

$$V\ (\text{max.}) = + \frac{P\frac{y}{4}}{y + \frac{l}{2}}$$

for the greatest tension, and

$$V\ (\text{min.}) = \frac{-R\left(\frac{y+l}{4}\right) - Q\left(\frac{y+l}{2}\right)}{y + \frac{l}{2}}$$

for the greatest compression.

The above is expressed by the following rule:—

Consider that the structure is fully loaded and form the equation of moments accordingly for the bar the greatest stresses in which are to be found. Arrange this equation so that the effect of each load can be easily ascertained. Then to find the greatest tension leave out all the temporary loads that diminish the stress and to find the least tension, or the greatest compression, leave out all the temporary loads that increase the stress.

Or shorter thus: *In the equation giving the greatest stress in a bar (either tension or compression) the members containing the moving loads must have the same sign.*

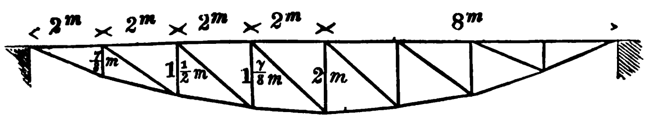
The equation of moments for the fully loaded bridge gives the greatest stress only in one case; when the members containing the moving loads have all the same sign.

The following numerical example will illustrate the above rule.

§ 6.—Parabolic Girder* of 16 Metres Span with a Single System of Diagonals.

The dimensions are given in Fig. 21.—The dead load on the bridge, designed for a single line of railway, can be taken at 1000 kilos. per metre and the live load at 5000 kilos. per

Fig. 21.



metre. One half of this is carried by each girder, and the length of each bay being 2 metres, 1000 kilos. dead load and 5000 kilos. live load act on each joint (Fig. 22).

To find the stress X, take a section $\alpha\beta$ through the first bay and form the equation of moments for the part shown in Fig. 23 round the point C.

$$0 = X_1 \times \tfrac{7}{8} + D \times 2.$$

* Thus called because the bow is in the form of a polygon inscribed in a parabola.—Trans.

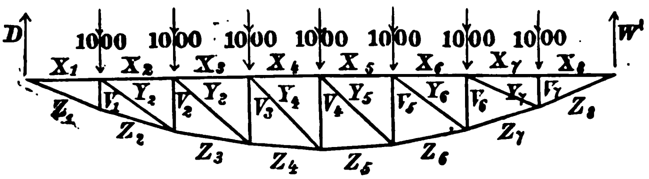
But when the bridge is fully loaded,

$$D = 1000\left(\tfrac{1}{8}+\tfrac{2}{8}+\tfrac{3}{8}+\tfrac{4}{8}+\tfrac{5}{8}+\tfrac{6}{8}+\tfrac{7}{8}\right) + 5000\left(\tfrac{1}{8}+\tfrac{2}{8}+\tfrac{3}{8}+\tfrac{4}{8}+\tfrac{5}{8}+\tfrac{6}{8}+\tfrac{7}{8}\right).$$

Therefore, substituting this value of D,

$$0 = X_1 \times \tfrac{7}{8} + 1000\left(\tfrac{1}{8}+\tfrac{2}{8}+\tfrac{3}{8}+\tfrac{4}{8}+\tfrac{5}{8}+\tfrac{6}{8}+\tfrac{7}{8}\right) \times 2 + 5000\left(\tfrac{1}{8}+\tfrac{2}{8}+\tfrac{3}{8}+\tfrac{4}{8}+\tfrac{5}{8}+\tfrac{6}{8}+\tfrac{7}{8}\right) \times 2.$$

FIG. 22.



It will be observed that the seven members of this equation due to the live load have all the same sign, and hence the greatest stress in the bar X_1 occurs when the bridge is fully loaded. Solving:—

$$X_1 \text{ (min.)} = -48000 \text{ kilos.}$$

The stress Z_1 can also be obtained from Fig. 23 by taking moments round B.

$$0 = -Z_1 \times 0{\cdot}8 + D \times 2,$$

or substituting for D its value.

$$0 = -Z_1 \times 0{\cdot}8 + 1000\left(\tfrac{1}{8}+\tfrac{2}{8}+\tfrac{3}{8}+\tfrac{4}{8}+\tfrac{5}{8}+\tfrac{6}{8}+\tfrac{7}{8}\right) \times 2 + 5000\left(\tfrac{1}{8}+\tfrac{2}{8}+\tfrac{3}{8}+\tfrac{4}{8}+\tfrac{5}{8}+\tfrac{6}{8}+\tfrac{7}{8}\right) \times 2.$$

FIG. 23.

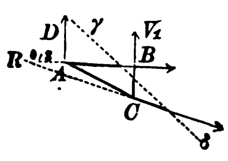
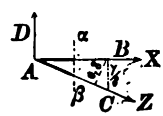


FIG. 24.



Here also the greatest stress occurs when the bridge is fully loaded, therefore

$$Z_1 \text{ (max.)} = +52500 \text{ kilos.}$$

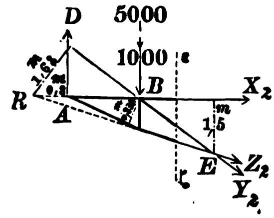
To find the stress V_1 take a section $\gamma\delta$ and form the equation of moments for the part shown in Fig. 24 round the point R.

$$0 = -V_1 \times 2{\cdot}8 - D \times 0{\cdot}8.$$

and substituting for D its value

$$0 = -\,V_1 \times 2{\cdot}8 - 1000\,(\tfrac{1}{8}+\tfrac{2}{8}+\tfrac{3}{8}+\tfrac{4}{8}+\tfrac{5}{8}+\tfrac{6}{8}+\tfrac{7}{8}) \times 2$$
$$- 5000\,(\tfrac{1}{8}+\tfrac{2}{8}+\tfrac{3}{8}+\tfrac{4}{8}+\tfrac{5}{8}+\tfrac{6}{8}+\tfrac{7}{8}) \times 2.$$

FIG. 25.



Here again it is evident that V_1 is greatest when the bridge is fully loaded. Hence

$$V_1 \text{ (min.)} = -\,6000 \text{ kilos.}$$

The stresses X_2, Y_2, Z_2 can be found by cutting off the part of the girder shown in Fig. 25. For X_2 take moments round E

$$0 = X_2 \times 1{\cdot}5 + D \times 4 - 1000 \times 2 - 5000 \times 2,$$

or substituting for D.

$$0 = X_2 \times 1{\cdot}5 + 1000\,(\tfrac{1}{8}+\tfrac{2}{8}+.+\tfrac{7}{8}) \times 4$$
$$+ 5000\,(\tfrac{1}{8}+\tfrac{2}{8}+.+\tfrac{7}{8}) \times 4$$
$$- 1000 \times 2 - 5000 \times 2.$$

The live load of 5000 kilos. acting at B is contained in two members of this equation. One, $+\,5000 \times \tfrac{7}{8} \times 4$, is the effect produced by the part of the load transmitted to the abutment A, and the other, $-\,5000 \times 2$ is the direct effect of the load. According to the rule these two members must be united into one, viz. $5000\,(\tfrac{7}{8} \times 4 - 2)$, the equation then takes the form :—

$$0 = X_2 \times 1{\cdot}5 + 1000\,\{(\tfrac{1}{8}+\ldots+\tfrac{6}{8})\,4 + (\tfrac{7}{8} \times 4 - 2)\}$$
$$+ 5000\,\{(\tfrac{1}{8}+\ldots+\tfrac{6}{8})\,4 + (\tfrac{7}{8} \times 4 - 2)\}.$$

It is easily seen that all the members multiplied by 5000 are positive, hence the greatest stress occurs where the bridge is fully loaded, and

$$X_2 \text{ (min.)} = -\,48000 \text{ kilos.}$$

To find Y_2 take moments round R and by substituting for D its value

$$0 = Y_2 \times 1{\cdot}68 - 1000\,(\tfrac{1}{8}+\ldots+\tfrac{7}{8})\,0{\cdot}8 - 5000\,(\tfrac{1}{8}+\ldots+\tfrac{7}{8}) \times 0{\cdot}8$$
$$+ 1000 \times 2{\cdot}8 + 5000 \times 2{\cdot}8;$$

or arranging the equation according to the rule,

$$0 = Y_2 \times 1{\cdot}68 - 1000\,\{(\tfrac{1}{8}+\ldots+\tfrac{6}{8})\,0{\cdot}8 - (2{\cdot}8 - \tfrac{7}{8}\,0{\cdot}8)\}$$
$$- 5000\,(\tfrac{1}{8}+\ldots+\tfrac{6}{8})\,0{\cdot}8 + 5000\,(2{\cdot}8 - \tfrac{7}{8}\,0{\cdot}8).$$

Of the seven members multiplied by 5000, and representing the effect of the moving load, 6 are negative and 1 is positive. Leaving out therefore the positive member (which diminishes the stress)

$$0 = Y_2 \times 1\cdot68 - 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right) 0\cdot8 - \left(2\cdot8 - \tfrac{7}{8}\, 0\cdot8\right) \right\} \\ - 5000 \left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right) 0\cdot8,$$

whence

$$Y_2 \text{ (max.)} = + 6250 \text{ kilos.}$$

Next leaving out the six negative members,

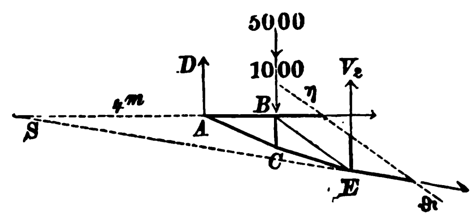
$$0 = Y_2 \times 1\cdot68 - 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right) 0\cdot8 - \left(2\cdot8 - \tfrac{7}{8}\, 0\cdot8\right) \right\} \\ + 5000 \left(2\cdot8 - \tfrac{7}{8}\, 0\cdot8\right),$$

whence

$$Y_2 \text{ (min.)} = - 6250 \text{ kilos.}$$

(It appears that $Y_2 = 0$ when the bridge is fully loaded. This result will be explained further on when treating of the theory of parabolic girders.)

Fig. 26.



The stress Z_2 is found by taking moments round B and arranging the equation as before.

$$0 = - Z_2 \times 0\cdot835 + 1000 \left(\tfrac{1}{8} + \ldots + \tfrac{7}{8}\right) \times 2 + 5000 \left(\tfrac{1}{8} + \ldots + \tfrac{7}{8}\right) \times 2,$$

from which it is evident that Z_2 is a maximum when the bridge is fully loaded. Hence

$$Z_2 \text{ (max.)} = + 50300 \text{ kilos.}$$

To determine V_2 take a section $\eta\,\theta$ and form the equation of moments for the part shown in Fig. 26 with S as turning point.

Arranging this equation

$$0 = -V_2 \times 8 - 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right) 4 - \left(6 - \tfrac{7}{8} . 4\right) \right\}$$
$$- 5000 \left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right) 4 + 5000 \left(6 - \tfrac{7}{8} \times 4\right).$$

First leaving out the positive member multiplied by 5000

$$0 = -V_2 \times 8 - 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right) 4 - \left(6 - \tfrac{7}{8} \times 4\right) \right\}$$
$$- 5000 \left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right) 4,$$

or

$$V_2 \text{ (min.)} = -7560 \text{ kilos.}$$

Then leaving out the negative members multiplied by 5000

$$0 = -V_2 \times 8 - 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right) 4 - \left(6 - \tfrac{7}{8} \times 4\right) \right\}$$
$$+ 5000 \left(6 - \tfrac{7}{8} \times 4\right),$$

or

$$V_2 \text{ (max.)} = +560 \text{ kilos.}$$

These examples sufficiently illustrate the rule, and the calculations for the remaining bars need not be given so fully.

The general equation of moments and the results for the remaining bars are given below.

$$0 = X_3 \times 1{\cdot}875 + 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 6 + \left(\tfrac{6}{8} . 6 - 2\right) + \tfrac{7}{8} . 6 - 4\right) \right\}$$
$$+ 5000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 6 + \left(\tfrac{6}{8} . 6 - 2\right) + \left(\tfrac{7}{8} . 6 - 4\right) \right\}$$
$$X_3 \text{ (min.)} = -48000 \text{ kilos.}$$

$$0 = Y_3 \times 5{\cdot}47 - 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 4 - \left(8 - \tfrac{6}{8} . 4\right) - \left(6 - \tfrac{7}{8} . 4\right) \right\}$$
$$- 5000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 4 + 5000 \left(8 - \tfrac{6}{8} \times 4\right) + 5000 \left(6 - \tfrac{7}{8} . 4\right) \right\}$$
$$Y_3 \begin{cases} \text{(max.)} = +6850 \text{ kilos.} \\ \text{(min.)} = -6850 \text{ kilos.} \end{cases}$$

$$0 = -Z_3 \times 1{\cdot}474 + 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right) 4 + \left(\tfrac{7}{8} . 4 - 2\right) \right\}$$
$$+ 5000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right) 4 + \left(\tfrac{7}{8} . 4 - 2\right) \right\}$$
$$Z_3 \text{ (max.)} = +48900 \text{ kilos.}$$

$$0 = -V_3 \times 30 - 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 24 - \left(28 - \tfrac{6}{8} \times 24\right) - \left(26 - \tfrac{7}{8} . 24\right) \right\}$$
$$- 5000 \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 24 + 5000 \left(28 - \tfrac{6}{8} . 24\right) + 5000 \left(26 - \tfrac{7}{8} . 24\right)$$
$$V_3 \begin{cases} \text{(max.)} = +1500 \text{ kilos.} \\ \text{(min.)} = -8500 \text{ kilos.} \end{cases}$$

$$0 = X_4 \times 2 + 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{4}{8}\right) 8 + \left(\tfrac{5}{8} . 8 - 2\right) + \left(\tfrac{6}{8} . 8 - 4\right) + \left(\tfrac{7}{8} . 8 - 6\right) \right\}$$
$$+ 5000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{4}{8}\right) 8 + \left(\tfrac{5}{8} . 8 - 2\right) + \left(\tfrac{6}{8} . 8 - 4\right) + \left(\tfrac{7}{8} . 8 - 6\right) \right\}$$
$$X_4 \text{ (min.)} = -48000 \text{ kilos.}$$

$$0 = Y_4 \times 21{\cdot}2 - 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{4}{8}\right) 24 - \left(30 - \tfrac{5}{8} . 24\right) - \left(28 - \tfrac{6}{8} . 24\right) \right.$$
$$\left. - \left(26 - \tfrac{7}{8} . 24\right) \right\}$$
$$+ 5000 \left(\tfrac{1}{8} + \ldots + \tfrac{4}{8}\right) 24 + 5000 \left\{ \left(30 - \tfrac{5}{8} . 24\right) + \left(28 - \tfrac{6}{8} . 24\right) \right.$$
$$\left. + \left(26 - \tfrac{7}{8} . 24\right) \right\}$$
$$Y_4 \begin{cases} \text{(max.)} = +7080 \text{ kilos.} \\ \text{(min.)} = -7080 \text{ kilos.} \end{cases}$$

$$0 = -Z_4 \times 1{\cdot}873 + 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 6 + \left(\tfrac{6}{8} \,.\, 6 - 2\right) + \left(\tfrac{7}{8} \,.\, 6 - 4\right) \right\}$$
$$+ 5000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 6 + \left(\tfrac{6}{8} \,.\, 6 - 2\right) + \left(\tfrac{7}{8} \,.\, 6 - 4\right) \right\}$$
$$Z_4 \text{ (max.)} = +\, 48100 \text{ kilos.}$$

(The following equations of moments are formed with reference to the part of the girder lying to the right of the section line):

$$0 = -V_4 \times 32 + 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{3}{8}\right) 24 - \left(32 - \tfrac{4}{8} \,.\, 24\right) - \left(30 - \tfrac{5}{8} \,.\, 24\right) \right.$$
$$\left. - \left(28 - \tfrac{6}{8} \,.\, 24\right) \left(-\, 26 - \tfrac{7}{8} \,.\, 24\right) \right\}$$
$$+ 5000 \left(\tfrac{1}{8} + \ldots + \tfrac{3}{8}\right) 24$$
$$- 5000 \left\{ \left(32 - \tfrac{4}{8} \,.\, 24\right) + \left(30 - \tfrac{5}{8} \,.\, 24\right) \right.$$
$$\left. + \left(28 - \tfrac{6}{8} \,.\, 24\right) + \left(26 - \tfrac{7}{8} \,.\, 24\right) \right\}$$
$$V_4 \begin{cases} \text{(max.)} = +\, 1800 \text{ kilos.} \\ \text{(min.)} = -\, 8800 \text{ kilos.} \end{cases}$$

$$0 = -X_5 \times 1{\cdot}875 - 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 6 + \left(\tfrac{6}{8} \,.\, 6 - 2\right) + \left(\tfrac{7}{8} \,.\, 6 - 4\right) \right\}$$
$$- 5000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 6 + \left(\tfrac{6}{8} \,.\, 6 - 2\right) + \left(\tfrac{7}{8} \,.\, 6 - 4\right) \right\}$$
$$X_5 \text{ (min.)} = -\, 48000.$$

$$0 = Y_5 \times 21{\cdot}88 + 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{4}{8}\right) 24 - \left(30 - \tfrac{5}{8} \,.\, 24\right) - \left(28 - \tfrac{6}{8} \,.\, 24\right) \right.$$
$$\left. - \left(26 - \tfrac{7}{8} \,.\, 24\right) \right\}$$
$$+ 5000 \left(\tfrac{1}{8} + \ldots + \tfrac{4}{8}\right) 24 - 5000 \left\{ \left(30 - \tfrac{5}{8} \,.\, 24\right) + \left(28 - \tfrac{6}{8} \,.\, 24\right) \right.$$
$$\left. - \left(26 - \tfrac{7}{8} \,.\, 24\right) \right\}$$
$$Y_5 \begin{cases} \text{(max.)} = +\, 6850 \text{ kilos.} \\ \text{(min.)} = -\, 6850 \text{ kilos.} \end{cases}$$

$$0 = Z_5 \times 1{\cdot}996 - 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{4}{8}\right) 8 + \left(\tfrac{5}{8} \,.\, 8 - 2\right) + \left(\tfrac{6}{8} \,.\, 8 - 4\right) + \left(\tfrac{7}{8} \,.\, 8 - 6\right) \right\}$$
$$- 5000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{4}{8}\right) 8 + \left(\tfrac{5}{8} \,.\, 8 - 2\right) + \left(\tfrac{6}{8} \,.\, 8 - 4\right) + \left(\tfrac{7}{8} \,.\, 8 - 6\right) \right\}$$
$$Z_5 \text{ (max.)} = +\, 48100 \text{ kilos.}$$

$$0 = -V_5 \times 10 + 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{4}{8}\right) 4 - \left(10 - \tfrac{5}{8} \,.\, 4\right) \right.$$
$$\left. - \left(8 - \tfrac{6}{8} \,!\, 4\right) - \left(6 - \tfrac{7}{8} \,.\, 4\right) \right\}$$
$$+ 5000 \left(\tfrac{1}{8} + \ldots + \tfrac{4}{8}\right) 4 - 5000 \left\{ \left(10 - \tfrac{5}{8} \,.\, 4\right) \right.$$
$$\left. + \left(8 - \tfrac{6}{8} \,.\, 4\right) + \left(6 - \tfrac{7}{8} \,.\, 4\right) \right\}$$
$$V_5 \begin{cases} \text{(max.)} = +\, 1500 \text{ kilos.} \\ \text{(min.)} = -\, 8500 \text{ kilos.} \end{cases}$$

$$0 = -X_6 \times 1{\cdot}5 - 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right) 4 + \left(\tfrac{7}{8} \,.\, 4 - 2\right) \right\}$$
$$- 5000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right) 4 + \left(\tfrac{7}{8} \,.\, 4 - 2\right) \right\}$$
$$X_2 \text{ (min.)} = -\, 48000 \text{ kilos.}$$

$$0 = Y_6 \times 6 + 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 4 - \left(8 - \tfrac{6}{8} \,.\, 4\right) - \left(6 - \tfrac{7}{8} \,.\, 4\right) \right\}$$
$$+ 5000 \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 4 - 5000 \left\{ \left(8 - \tfrac{6}{8} \,.\, 4\right) + \left(6 - \tfrac{7}{8} \,.\, 4\right) \right\}$$
$$Y_6 \begin{cases} \text{(max.)} = +\, 6250 \text{ kilos.} \\ \text{(min.)} = -\, 6250 \text{ kilos.} \end{cases}$$

$$0 = Z_6 \times 1{\cdot}84 - 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 6 + \left(\tfrac{6}{8} \,.\, 6 - 2\right) + \left(\tfrac{7}{8} \,.\, 6 - 4\right) \right\}$$
$$- 5000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 6 + \left(\tfrac{6}{8} \,.\, 6 - 2\right) + \left(\tfrac{7}{8} \,.\, 6 - 4\right) \right\}$$
$$Z_6 \text{ (max.)} = +\, 48900 \text{ kilos.}$$

$$0 = -V_6 \times 4{\cdot}8 + 1000 \left\{ \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 0{\cdot}8 - \left(4{\cdot}8 - \tfrac{6}{8} \,.\, 0{\cdot}8\right) - \left(2{\cdot}8 - \tfrac{7}{8} \,.\, 0{\cdot}8\right) \right\}$$
$$+ 5000 \left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) 0{\cdot}8 - 5000 \left\{ \left(4{\cdot}8 - \tfrac{6}{8} \,.\, 0{\cdot}8\right) \right.$$
$$\left. + \left(2{\cdot}8 - \tfrac{7}{8} \,.\, 0{\cdot}8\right) \right\}$$
$$V_6 \begin{cases} \text{(max.)} = +\, 560 \text{ kilos.} \\ \text{(min.)} = -\, 7560 \text{ kilos.} \end{cases}$$

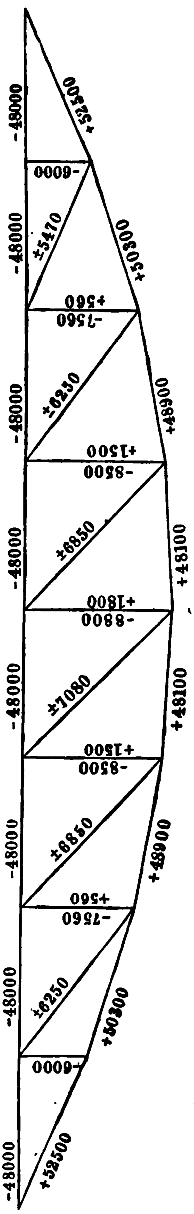


FIG. 27.

$$0 = -X_7 \times 0{\cdot}875 - 1000\left(\tfrac{1}{8} + \dots + \tfrac{7}{8}\right)2 - 5000\left(\tfrac{1}{8} + \dots + \tfrac{7}{8}\right)2$$

X_7 (min.) $= -$ 48000 kilos.

$$0 = Y_7 \times 1{\cdot}92 + 1000\left\{\left(\tfrac{1}{8} + \dots + \tfrac{6}{8}\right)0{\cdot}8 - \left(2{\cdot}8 - \tfrac{7}{8}\,.\,0{\cdot}8\right)\right\} + 5000\left(\tfrac{1}{8} + \dots + \tfrac{6}{8}\right)0{\cdot}8 - 5000\left(2{\cdot}8 - \tfrac{7}{8}\,.\,0{\cdot}8\right)$$

$Y_7 \begin{cases} \text{(max.)} = +\ 5470 \text{ kilos.} \\ \text{(min.)} = -\ 5470 \text{ kilos.} \end{cases}$

$$0 = Z_7 \times 1{\cdot}43 + 1000\left\{\left(\tfrac{1}{8} + \dots + \tfrac{6}{8}\right)4 + \left(\tfrac{7}{8}\,.\,4 - 2\right)\right\} - 5000\left\{\left(\tfrac{1}{8} + \dots + \tfrac{6}{8}\right)4 + \left(\tfrac{7}{8}\,.\,4 - 2\right)\right\}$$

Z_7 (max.) $= +$ 50300 kilos.

$$0 = -V_7 \times 2 - 1000 \times 2 - 5000 \times 2$$

V_7 (min.) $= -$ 6000 kilos.

$$0 = -X_8 \times 0{\cdot}875 - 1000\left(\tfrac{1}{8} + \dots + \tfrac{7}{8}\right)2 - 5000\left(\tfrac{1}{8} + \dots + \tfrac{7}{8}\right)2$$

X_8 (min.) $= -$ 48000 kilos.

$$0 = Z_8 \times 0{\cdot}8 - 1000\left(\tfrac{1}{8} + \dots + \tfrac{7}{8}\right)2 - 5000\left(\tfrac{1}{8} + \dots + \tfrac{7}{8}\right)2$$

Z_8 (max.) $= -$ 52500 kilos.

These results are shown in Fig 27.

§ 7.—DERIVED FORMS.

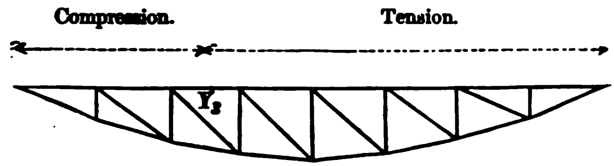
From the above calculations it is apparent that the greatest stresses in the vertical and diagonal braces occur when the bridge is partially loaded. It will be interesting to ascertain according to what law the girder is loaded when the greatest stresses obtain in the braces. By noticing in each case what temporary loads are left out of the general equation of moments it will be observed that any diagonal brace, Y_3 for instance, will be subject to the greatest tension when all the joints lying to the right of it are loaded, and will be under the greatest compression when all the joints lying to the left are loaded. This is represented in Fig. 28 by the words "Tension" and "Compression."

Evidently if this diagonal were inclined upward to the right instead of to the left the

words in Fig. 28 would simply have to change places, and also if the girder be looked at from behind (or else its image in a looking-glass) the diagonal Y_6 will appear in the same bay as Y_3 in Fig. 28; thus the arrangement of the moving load to produce the greatest stresses in Y_6 will be as shown in Fig. 29.

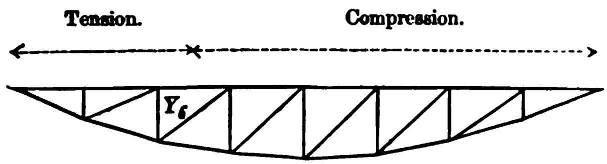
If both diagonals are present in the same girder, as shown in Fig. 30, and are so constructed as to be incapable of resisting

FIG. 28.



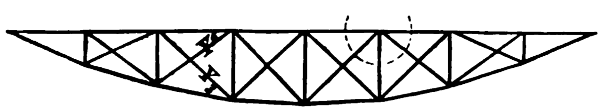
compression, they will come into play only when the loading is such as to produce tension in them; at other times they will be subject to no stress just as if they were threads. In such a

FIG. 29.



girder therefore only the greatest tension given for the diagonal bars in the above example need be considered. Thus in Fig. 30 the greatest tension in the diagonals of the third bay from the

FIG. 30.



left will be for the brace inclined upwards to the left the same as that in Y_3 (as found in the previous numerical example), and for the brace inclined upward to the right the same as that in Y_6. Similarly the greatest tension in the other diagonal braces of Fig. 30 can be written down from Fig. 27.

The vertical braces are always in compression in this case,

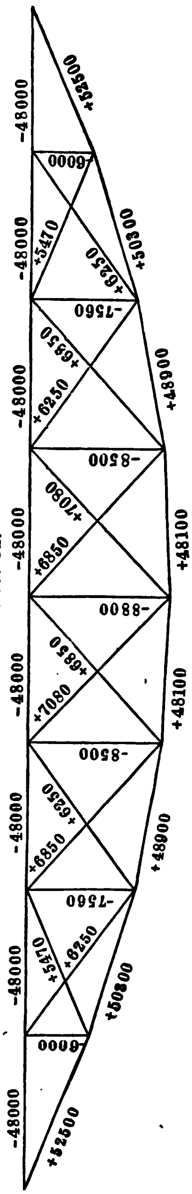
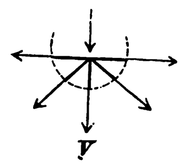


Fig. 32.

as will appear at once from Fig. 31, for since the diagonals are incapable of resisting compression there would be nothing to oppose the vertical downward force produced by the vertical brace if it were in tension. The greatest compression in the verticals will be the same as given in Fig. 27, for only one of the diagonals in each bay is in tension at a time, and the other being therefore slack can be considered as absent.

Fig. 31.



Thus without any further calculations the greatest stresses in a girder with crossed diagonals can be written down from those obtained in the previous example, and this is done in Fig. 32.

If the diagonals are so constructed that they can only take up compression (this is sometimes the case in wooden girders), it will appear by similar reasoning that for the diagonals, only the greatest compression, and for the verticals only the greatest tension, found in the previous example, will apply. As regards the minimum stress or compression in the verticals, the load each vertical supports at the top joint can alone produce compression in it, for the diagonals cannot do so, as they never can be in tension. This load is either 1000 kil. or 1000 + 5000 kil., and therefore the greatest compression in the verticals is

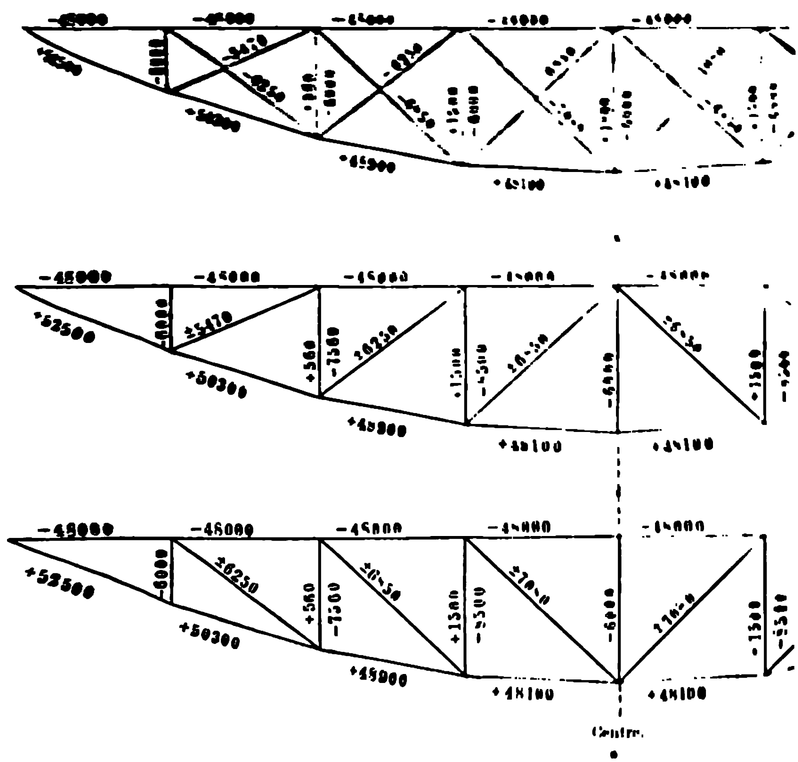
$$V \text{ (min.)} = -6000 \text{ kilos.}$$

The greatest stresses in a girder of the above construction

are given in Fig. 33, and the diagonals are shown in double lines to express their incapability to resist tension.

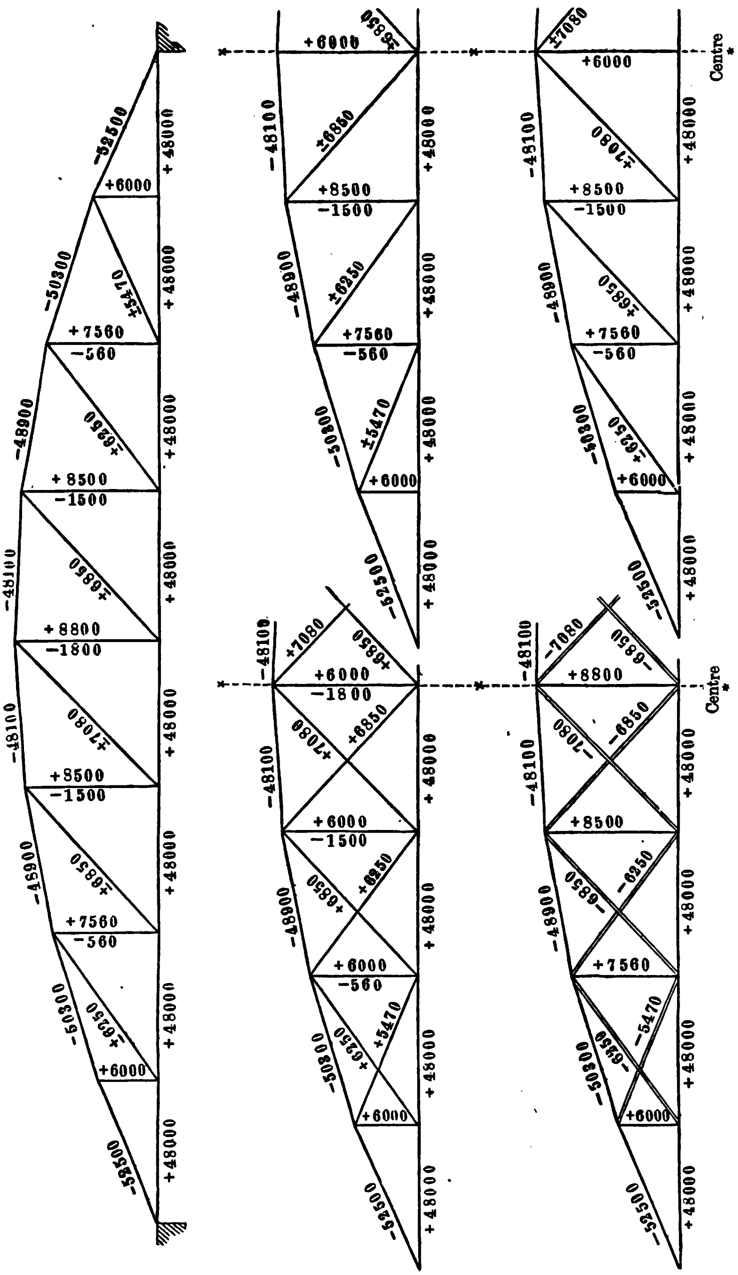
In girders with a single system of diagonals varying, however, from Fig. 27 in that the arrangement is symmetrical on each side of the centre, the greatest stresses can be written

FIGS. 33, 34, AND 35.



down at once from the above, with the single exception of that in the central vertical brace.

The stress in the central vertical of Fig. 34 evidently depends on the tension in the adjacent parts of the lower boom at its foot, and it must therefore always be in compression. This compression will reach its greatest value when the tension in the boom is a maximum, that is, when the bridge is fully



FIGS. 36, 37, 38, 39, AND 40.

loaded; and in this case each vertical brace has a compression of 6000 kilos. to bear. Therefore, for the central vertical also,

$$V\ (\text{min.}) = -\ 6000\ \text{kilos.}$$

In Fig. 35 it is obvious that the central vertical can only be in a state of stress when there is a direct load on the top joint; this stress must therefore be compression, and its greatest value is evidently

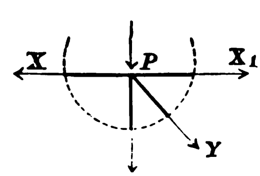
$$V\ (\text{min.}) = -\ 6000\ \text{kilos.}$$

Lastly, if in the girder shown in Fig. 27 the signs of all the stresses be changed, the greatest stresses for a parabolic girder having the bow above, as shown in Fig. 36, will be obtained. In fact, the whole of the reasoning and the equations are precisely the same, except that all the signs must be changed, and that maximum must be put for minimum, and minimum for maximum. The derived forms shown in Figs. 37, 38, 39, 40, can be obtained from Fig. 36, as before.

§ 8.—Theory of Parabolic Girders.

It appears from the above example that the stresses in a parabolic girder can be found by the method of moments, even when the theory of such girders is not known. Two properties of these girders were discovered: the first is that the stress in the horizontal boom is greatest when the bridge is fully loaded, and is then equal throughout; and the second, that when the bridge is fully loaded the stress in the diagonal braces is everywhere nil. The last property is in reality contained in the first, for when $X = X_1$ (Fig. 41), $Y = O$, or else the horizontal forces at P would not be in equilibrium.

Fig. 41.

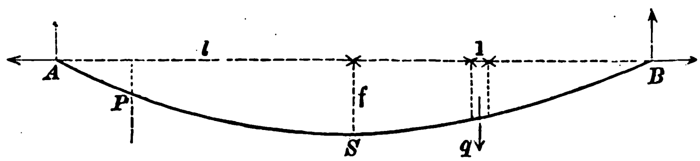


It will be useful to investigate the conditions upon which these properties depend. This knowledge is not necessary to enable the calculations for any given parabolic

girder to be made, but is required when the form of a girder is to be found which will have these properties.

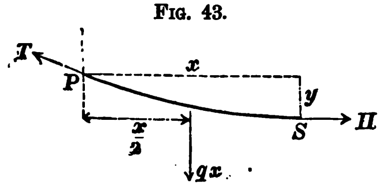
Consider a chain attached to the two fixed points A and B (Fig. 42), and hanging in its curve of equilibrium. Let the load be uniformly distributed over the span AB, and equal to q per unit of length. Suppose that the chain is cut at its

FIG. 42.



lowest point S (where it is horizontal), and a horizontal force H applied at the point of section to maintain equilibrium. This force must be horizontal, since the part of the chain at S is horizontal. Let the chain be also cut at any other point P, applying a force T to maintain equilibrium. It is evident that this force must lie in the direction of the tangent at the point P.

FIG. 43.



The piece S P of the chain (Fig. 43) is held in equilibrium by three forces: viz., H, T and the resultant of the load on the part S P. This last force is equal to $q x$, where x denotes the horizontal distance of P from S; and its point of application is at a distance $\frac{x}{2}$ from either P or S, since the load is uniformly distributed over x.

Taking moments round P:—

$$\mathrm{H} y = q x \cdot \frac{x}{2}. \tag{1}$$

But since P is any point on the curve, this equation is true for the point A, thus substituting l for x, and f for y.

$$\mathrm{H} f - q l \cdot \frac{l}{2}. \tag{2}$$

Dividing *eq*. (1) by *eq*. (2).

$$\frac{y}{f} = \frac{x^2}{l^2}. \qquad (3)$$

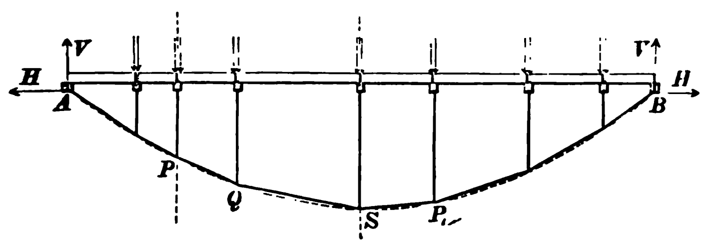
This is the equation to a parabola, and it is evidently also applicable to the part S B of the chain. Thus the position of all points of the chain can be determined by this equation, by giving values to x and solving for y.

It is evident from Fig. 43 that the horizontal component of T is everywhere equal to H, it is so therefore at the points of attachment A and B; it is also evident that the vertical component V of T is $q.x$, and therefore equal to $q.l$ at A and B; and, lastly, that

$$T = \sqrt{H^2 + V^2}.$$

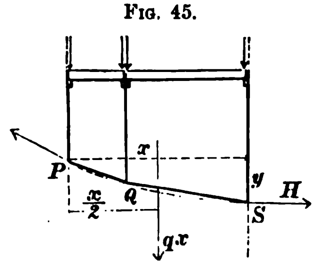
If the manner of loading is altered, some points of the chain

FIG. 44.



may still remain on the parabola, and for these points equation (3) will hold good. This is the case, for instance, when the loads on each side of S are concentrated at points, so long as the load at each point is equal to the sum of half the distributed load on the two adjacent bays (Fig. 44); for the part S P of the chain (Fig. 45) will still be subject to the vertical load $q.x$ (the resultant of the four concentrated loads shown in the figure), and the point of application of this resultant will still be at the distance $\frac{x}{2}$ from S.

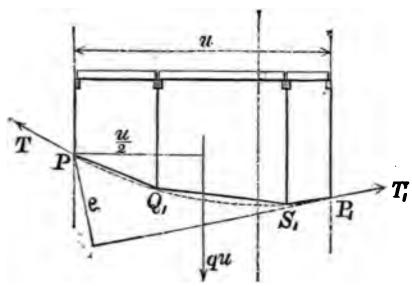
FIG. 45.



The chain can be loaded as above by employing vertical struts to transfer the uniformly distributed load (Fig. 44). The unloaded portions of the chain are evidently straight, and the form of equilibrium will therefore be that of a polygon inscribed in a parabola. The above is even true if the vertex is not a loaded point; for consider the part P P_1 of the chain cut out, and equilibrium maintained by the forces T,T_1 (Fig. 46). Taking moments round P,

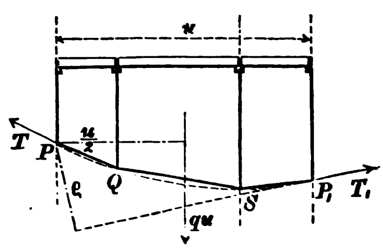
$$T_1\,\rho = q\,u\,.\,\frac{u}{2};$$

FIG. ~~46~~. 47



and this equation will not be altered if, instead of S and Q, other points, such as S_1 and Q_1 (Fig. 47), are taken as loaded points.

FIG. ~~47~~. 46



If the two points A and B cannot offer any horizontal resistance but only vertical reactions, other means to resist the horizontal pull H must be adopted, for instance, introducing a horizontal boom. Thus a parabolic girder of the form shown in Fig. 48 is obtained, which can carry a load, uniformly distri-

buted over the whole span, without requiring any diagonal braces. The conditions, therefore, that must obtain, in order that the girder may have the properties mentioned above, can be briefly stated thus:—

The feet of the verticals must lie in a parabola, the axis of which is vertical and passes through the centre of the span.

FIG. 48.

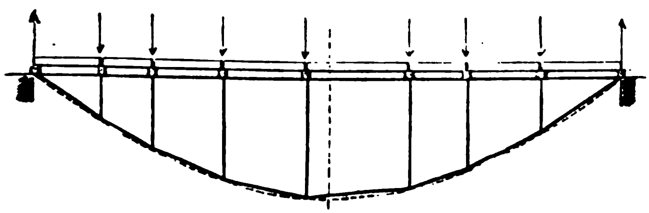
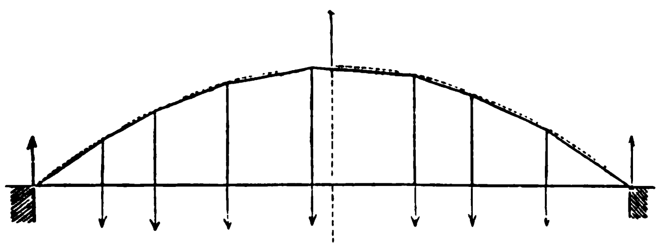


FIG. 49.



The whole of the above reasoning remains true, if the girder be turned upside down and all the forces reversed (Fig. 49); it need not, therefore, be repeated.

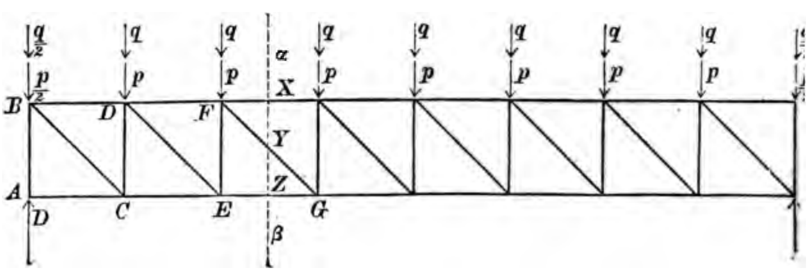
This subject will be further considered under the head of "Sickle-shaped Trusses" (Bowstring-roofs).

THIRD CHAPTER.

§ 9.—Application of the Method of Moments to the Calculation of the Stresses in Braced Girders having Parallel Booms.

The method of moments can also be employed in calculating ordinary braced girders divided into rectangular bays. It is hardly necessary to observe that the equation of moments remains true although two of the three bars cut through are parallel, their point of intersection being therefore at an infinite distance, and consequently the lever arm of the stress in the third bar being also infinite. All the lever arms in the equation of moments are, however, infinite, and thus divide out of the equation, enabling the required stress to be determined.* For example, in the girder shown in Fig. 50, the stress Y in the diagonal F G is to be found by taking a section $\alpha\,\beta$, applying the forces X, Y, Z to maintain equilibrium, and forming the

Fig. 50.

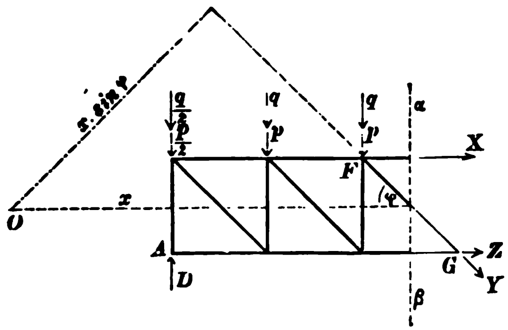


equation of moments for one of the parts of the girder (Fig. 51) with reference to the point of intersection of X and

* In this case these infinities are all equal; they can therefore be considered a common factor, but generally it is a mathematical fallacy to treat infinite s in this manner.—Trans.

Z. This point, which is at infinity, can be considered as lying on the central horizontal line of the girder, and since X and Z pass through it their lever arms are *nil*, but the lever arms of all the vertical forces are evidently infinite. Now, if O were the point of intersection of X and Z, at a distance x from the point where $\alpha\beta$ cuts Y, the lever arm of Y would be $x \,.\, \sin\phi$, and if O be considered to move off to infinity, x becomes infinite, and the lever arm of Y is $\infty \,.\, \sin\phi$.

FIG. 51.



Therefore, the equation of moments is

$$0 = \mathrm{Y}\,.\,\infty\,.\,\sin\phi - \mathrm{D}\,.\,\infty + \left(\frac{p}{2}+\frac{q}{2}\right)\infty + (p+q)\,\infty + (p+q)\infty\,.$$

or dividing out by ∞.

$$0 = \mathrm{Y}\,.\,\sin\phi - \mathrm{D} + \left(\frac{p}{2}+\frac{q}{2}\right) + (p+q) + (p+q).$$

Here $\mathrm{Y}\,.\,\sin\phi$ is the vertical component of Y, hence the above equation is merely the expression of the law that for equilibrium the sum of the vertical forces must be zero. Thus the principal object of the method of moments (to obtain an equation containing only one unknown) is, in this special case, arrived at by resolving the forces vertically. This shows the general applicability of the method of moments; for even in special cases like the present, in which a shorter way of obtaining the required result exists, it can be used, and even points to the shorter method.

Substituting the value of D in the above equation, viz.:

$$D = (p+q)\left\{\tfrac{1}{8} + \tfrac{2}{8} + \tfrac{3}{8} + \tfrac{4}{8} + \tfrac{5}{8} + \tfrac{6}{8} + \tfrac{7}{8} + \tfrac{1}{2}\right\},$$

and arranging it according to the rule, to ascertain the effect of the moving load:

$$\begin{aligned} 0 = \pm Y \,.\, \sin\phi - p\left\{\tfrac{1}{8} + \tfrac{2}{8} + \tfrac{3}{8} + \tfrac{4}{8} + \tfrac{5}{8} - (1 - \tfrac{6}{8}) - (1 - \tfrac{7}{8})\right\} \\ - q\,(\tfrac{1}{8} + \tfrac{2}{8} + \tfrac{3}{8} + \tfrac{4}{8} + \tfrac{5}{8}) + q\left\{(1 - \tfrac{6}{8}) + (1 - \tfrac{7}{8})\right\}. \end{aligned}$$

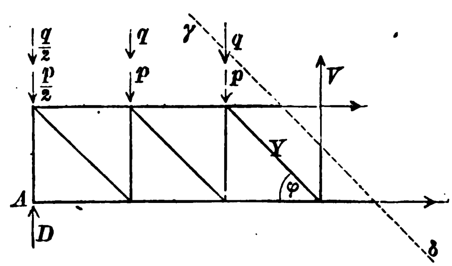
The maximum value of Y can now be obtained by leaving out the positive member produced by q, and the minimum value by omitting the negative member.

There is no difficulty in determining the stresses X and Z (Fig. 51). Let λ be the length of a bay, and h the height of the girder, take moments first round G and then round F, thus:

$$\begin{aligned} 0 &= Xh + (p+q)\left\{(\tfrac{1}{8} + \tfrac{2}{8} + \tfrac{3}{8} + \tfrac{4}{8} + \tfrac{5}{8}) \,.\, 3\lambda + (\tfrac{6}{8} \,.\, 3\lambda - \lambda) \right. \\ &\quad \left. + (\tfrac{7}{8} \,.\, 3\lambda - 2\lambda)\right\} \\ 0 &= -Zh + (p+q)\left\{(\tfrac{1}{8} + \tfrac{2}{8} + \tfrac{3}{8} + \tfrac{4}{8} + \tfrac{5}{8} + \tfrac{6}{8}) \,.\, 2\lambda + (\tfrac{7}{8} \,.\, 2\lambda - \lambda)\right\}. \end{aligned}$$

From which it is evident that these bars are in the greatest state of stress when the bridge is fully loaded.

FIG. 52.



Lastly, to find the stress in the adjacent vertical to the right, take a section $\gamma\,\delta$ (Fig. 52). The point about which to take moments is at infinity, and

$$0 = -V \,.\, \infty - D \,.\, \infty + \left(\frac{p}{2} + \frac{q}{2}\right)\infty + (p+q)\,\infty + (p+q)\,\infty.$$

The only difference between this equation and the one previously obtained for Y.sin ϕ is that —V is written instead of Y.sin ϕ. Hence it is obvious that

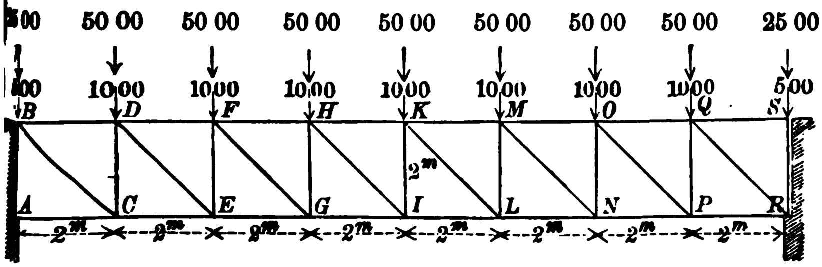
$$0 = -V - p\left\{\tfrac{1}{8} + \tfrac{2}{8} + \tfrac{3}{8} + \tfrac{4}{8} + \tfrac{5}{8} - (1 - \tfrac{6}{8}) - (1 - \tfrac{7}{8})\right\}$$
$$- q(\tfrac{1}{8} + \tfrac{2}{8} + \tfrac{3}{8} + \tfrac{4}{8} + \tfrac{5}{8}) + q\left\{(1 - \tfrac{6}{8}) - (1 - \tfrac{7}{8})\right\}.$$

The values Y.sin ϕ and —V are therefore identical; hence by first finding V, Y can be ascertained by dividing by (—sin ϕ). This is expressed in the following rule: *the resolved parts vertically of the stresses in a diagonal and vertical meeting at an unloaded joint are of equal magnitude, but of unlike sign.*

§ 10.—Braced Girder, of 16 Metres Span, composed of Single Right-angle Triangles.

Apart from the difference of form, the dimensions of this girder are the same as those of the parabolic girder, already calculated (p. 20), that is, the span (16m.) and the depth (2m.) are the same. The loads are also the same, viz., 1000 kilos. dead load and 5000 kilos. live load, on each bay. It is also assumed that the line of railway is on a level with the upper boom; these loads, therefore, act at the upper apices (Fig. 53).

Fig. 53.



Calculation of V_0 *and* Z_1.

Since V_0 and D are the only vertical forces acting at A (Fig. 54):

$$V_0 + D = 0, \quad \text{or} \quad V_0 = -D.$$

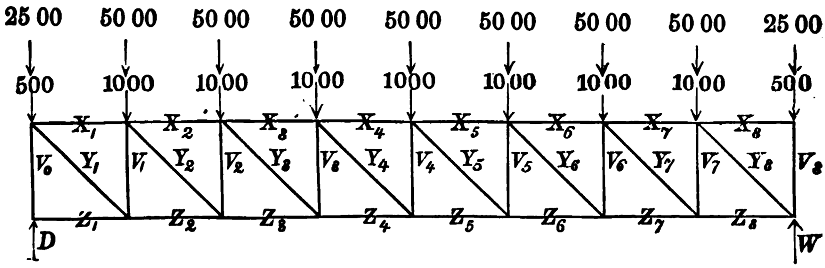
V_0 therefore reaches its greatest numerical value when D is greatest; that is evidently when the bridge is fully loaded. In this case $D = \frac{48000}{2}$ kilos., and hence:

$$V_0 \text{ (min.)} = -24000 \text{ kilos.}$$

Further, since Z_1 is the only horizontal force at A:

$$Z_1 = 0.$$

Fig. 54.



Calculation of X_8 *and* V_8.

Two vertical forces act at S (Fig. 53), viz., the load at this point (the greatest value of which is 3000 kilos.) and V_8. Therefore,

$$V_8 \text{ (min.)} = -3000 \text{ kilos.}$$

And since X_8 is the only horizontal force at S:

$$X_8 = 0.$$

Calculation of X_1, Z_2, V_1, Y_1.
(Section $a\beta$. Fig. 55.)

Taking moments for the part of the girder shown in Fig. 55, about the point C:

$$0 = X_1 \times 2 + (1000 + 5000) \times (\tfrac{1}{8} + \tfrac{2}{8} + \ldots + \tfrac{7}{8}) \times 2,$$

whence

$$X_1 \text{ (min.)} = -21000 \text{ kilos.}$$

The equation of moments about the point D is

$$0 = -Z_2 \times 2 + (1000 + 5000)\left(\tfrac{1}{8} + \ldots + \tfrac{7}{8}\right) 2.$$

Therefore,

$$Z_2 \text{ (max.)} = +21000 \text{ kilos.}$$

Resolving the forces vertically:

$$0 = -V_1 - 1000\left(\tfrac{1}{8} + \ldots + \tfrac{7}{8}\right) - 5000\left(\tfrac{1}{8} + \ldots + \tfrac{7}{8}\right),$$

or,

$$V_1 \text{ (min.)} = -21000 \text{ kilos.}$$

The diagonal Y_1 makes an angle of 45° with the horizontal. Therefore, the resolved part of Y_1 vertically is $Y_1 \,.\, \sin 45°$, or $Y_1 \,.\, \frac{1}{\sqrt{2}}$. Hence, by the rule:

$$Y_1 \,.\, \frac{1}{\sqrt{2}} = -V_1,$$

or

$$Y_1 = +21000 \times \sqrt{2},$$

$$Y_1 \text{ (max.)} = +29700 \text{ kilos.}$$

FIG. 55.

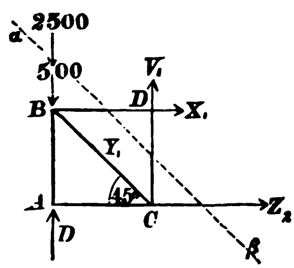
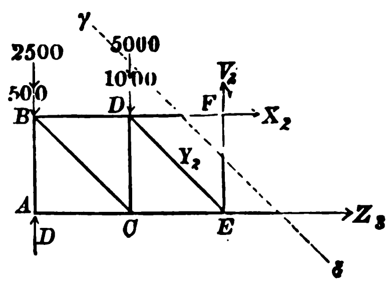


FIG. 56.



Calculation of X_2, Z_3, V_2, Y_2.
(*Section* $\gamma\delta$. *Fig.* 56.)

Taking moments about E:

$$0 = X_2 \times 2 + (1000 + 5000)\left\{\left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right) 4 + \left(\tfrac{7}{8} \,.\, 4 - 2\right)\right\},$$

or,

$$X_2 \text{ (min.)} = -36000 \text{ kilos.}$$

Taking moments about F :

$$0 = -Z_3 \times 2 + (1000 + 5000)\left\{\left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right)4 + \left(\tfrac{7}{8}\,.\,4 - 2\right)\right\}$$
$$Z_3\,(\text{max.}) = +\,36000 \text{ kilos.}$$

Resolving the forces vertically:

$$0 = -V_2 - 1000\left\{\tfrac{1}{8} + \ldots + \tfrac{6}{8} - \left(1 - \tfrac{7}{8}\right)\right\}$$
$$-\,5000\left(\tfrac{1}{8} + \ldots + \tfrac{6}{8}\right) + 5000\left(1 - \tfrac{7}{8}\right).$$

By leaving out of this equation, first the negative, then the positive members, multiplied by 5000 :

$$V_2\,(\text{max.}) = -\,1875 \text{ kilos.}$$
$$V_2\,(\text{min.}) = -\,15625 \text{ kilos.}$$

Then multiplying these values by $\sqrt{2}$, and changing the sign:

$$Y_2\,(\text{max.}) = +\,22100 \text{ kilos.}$$
$$Y_2\,(\text{min.}) = +\,2650 \text{ kilos.}$$

In a similar manner the stress in the remaining bars can be ascertained from the following equations:

$$0 = X_3 \times 2 + (1000 + 5000)\left\{\left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right)6 + \left(\tfrac{6}{8}\,.\,6 - 2\right) + \left(\tfrac{7}{8}\,.\,6 - 4\right)\right\}$$
$$X_3\,(\text{min.}) = -\,45000 \text{ kilos.}$$

$$0 = -Z_4 \times 2 + (1000 + 5000)\left\{\left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right)6 + \left(\tfrac{6}{8}\,.\,6 - 2\right) + \left(\tfrac{7}{8}\,.\,6 - 4\right)\right\}$$
$$Z_4\,(\text{max.}) = +\,45000 \text{ kilos.}$$

$$0 = -V_3 - 1000\left\{\tfrac{1}{8} + \ldots + \tfrac{5}{8} - \left(1 - \tfrac{6}{8}\right) - \left(1 - \tfrac{7}{8}\right)\right\}$$
$$-\,5000\left(\tfrac{1}{8} + \ldots + \tfrac{5}{8}\right) + 5000\left\{\left(1 - \tfrac{6}{8}\right) + \left(1 - \tfrac{7}{8}\right)\right\}$$

$$V_3\begin{cases}(\text{max.}) = +\quad 375 \text{ kilos.}\\(\text{min.}) = -\,10875 \text{ kilos.}\end{cases}$$

$$Y_3\begin{cases}(\text{max.}) = +\,15400 \text{ kilos.}\\(\text{min.}) = -\quad 530 \text{ kilos.}\end{cases}$$

$$0 = X_4 \times 2 + (1000 + 5000)\left\{\left(\tfrac{1}{8} + \ldots + \tfrac{4}{8}\right)8 + \left(\tfrac{5}{8}\,.\,8 - 2\right)\right.$$
$$\left. + \left(\tfrac{6}{8}\,.\,8 - 4\right) + \left(\tfrac{7}{8}\,.\,8 - 6\right)\right\}$$
$$X_4\,(\text{min.}) = -\,48000 \text{ kilos.}$$

$$0 = -Z_5 \times 2 + (1000 + 5000)\left\{\left(\tfrac{1}{8} + \ldots + \tfrac{4}{8}\right).\,8 + \left(\tfrac{5}{8}\,.\,8 - 2\right)\right.$$
$$\left. + \left(\tfrac{6}{8}\,.\,8 - 4\right) + \left(\tfrac{7}{8}\,.\,8 - 6\right)\right\}$$
$$Z_5\,(\text{max.}) = +\,48000 \text{ kilos.}$$

$$0 = -V_4 - 1000\left\{\tfrac{1}{8} + \ldots + \tfrac{4}{8} - \left(1 - \tfrac{5}{8}\right) - \left(1 - \tfrac{6}{8}\right) - \left(1 - \tfrac{7}{8}\right)\right\}$$
$$-\,5000\left(\tfrac{1}{8} + \ldots + \tfrac{4}{8}\right) + 5000\left\{\left(1 - \tfrac{5}{8}\right) + \left(1 - \tfrac{6}{8}\right) + \left(1 - \tfrac{7}{8}\right)\right\}$$

$$V_4\begin{cases}(\text{max.}) = +\,3250 \text{ kilos.}\\(\text{min.}) = -\,6750 \text{ kilos.}\end{cases}$$

$$Y_4\begin{cases}(\text{max.}) = +\,9550 \text{ kilos.}\\(\text{min.}) = -\,4600 \text{ kilos.}\end{cases}$$

The following equations are formed with reference to the part of the girder situated to the right of the section line:

$$0 = -X_5 \times 2 - (1000 + 5000)\left\{(\tfrac{1}{8} + \ldots + \tfrac{5}{8})\,6 + (\tfrac{6}{8}.6 - 2) + (\tfrac{7}{8}.6 - 4)\right\}$$
$$X_5\ (\text{min.}) = -45000 \text{ kilos.}$$

$$0 = Z_6 \times 2 - (1000 + 5000)\left\{(\tfrac{1}{8} + \ldots + \tfrac{5}{8})\,6 + (\tfrac{6}{8}.6 - 2) + (\tfrac{7}{8}.6 - 4)\right\}$$
$$Z_6\ (\text{max.}) = +45000 \text{ kilos.}$$

$$0 = -V_5 + 1000\left\{\tfrac{1}{8} + \ldots + \tfrac{4}{8} - (1 - \tfrac{5}{8}) - (1 - \tfrac{6}{8}) - (1 - \tfrac{7}{8})\right\}$$
$$+\,5000\,(\tfrac{1}{8} + \ldots + \tfrac{4}{8}) - 5000\left\{(1 - \tfrac{5}{8}) + (1 - \tfrac{6}{8}) + (1 - \tfrac{7}{8})\right\}$$
$$V_5 \begin{cases} (\text{max.}) = +6750 \text{ kilos.} \\ (\text{min.}) = -3250 \text{ kilos.} \end{cases}$$
$$Y_5 \begin{cases} (\text{max.}) = +4600 \text{ kilos.} \\ (\text{min.}) = -9550 \text{ kilos.} \end{cases}$$

$$0 = -X_6 \times 2 - (1000 + 5000)\left\{(\tfrac{1}{8} + \ldots + \tfrac{6}{8})\,4 + (\tfrac{7}{8}.4 - 2)\right\}$$
$$X_6\ (\text{min.}) = -36000 \text{ kilos.}$$

$$0 = Z_7 \times 2 - (1000 + 5000)\left\{(\tfrac{1}{8} + \ldots + \tfrac{6}{8})\,4 + (\tfrac{7}{8}.4 - 2)\right\}$$
$$Z_7\ (\text{max.}) = +36000 \text{ kilos.}$$

$$0 = -V_6 + 1000\left\{\tfrac{1}{8} + \ldots + \tfrac{5}{8} - (1 - \tfrac{6}{8}) - (1 - \tfrac{7}{8})\right\}$$
$$+\,5000\,(\tfrac{1}{8} + \ldots + \tfrac{5}{8}) - 5000\left\{(1 - \tfrac{6}{8}) + (1 - \tfrac{7}{8})\right\}$$
$$V_6 \begin{cases} (\text{max.}) = +10875 \text{ kilos.} \\ (\text{min.}) = -\ \ 375 \text{ kilos.} \end{cases}$$
$$Y_6 \begin{cases} (\text{max.}) = +\ \ 530 \text{ kilos.} \\ (\text{min.}) = -15400 \text{ kilos.} \end{cases}$$

$$0 = -X_7 \times 2 - (1000 + 5000)\,(\tfrac{1}{8} + \ldots + \tfrac{7}{8})\,.\,2$$
$$X_7\ (\text{min.}) = -21000 \text{ kilos.}$$

$$0 = Z_8 \times 2 - (1000 + 5000)\,(\tfrac{1}{8} + \ldots + \tfrac{7}{8})\,.\,2$$
$$Z_8\ (\text{max.}) = +21000 \text{ kilos.}$$

$$0 = -V_7 + 1000\left\{\tfrac{1}{8} + \ldots + \tfrac{6}{8} - (1 - \tfrac{7}{8})\right\} + 5000\,(\tfrac{1}{8} + \ldots + \tfrac{6}{8}) - 5000\,(1 - \tfrac{7}{8})$$
$$V_7 \begin{cases} (\text{max.}) = +15625 \text{ kilos.} \\ (\text{min.}) = +\ \ 1875 \text{ kilos.} \end{cases}$$
$$Y_7 \begin{cases} (\text{max.}) = -\ \ 2650 \text{ kilos.} \\ (\text{min.}) = -22100 \text{ kilos.} \end{cases}$$

The diagonal Y_8 does not meet any vertical at an unloaded joint, for the joint R (Fig. 53) cannot be considered unloaded on account of the reaction of the abutment. The rule for finding Y is therefore not applicable in this case. The vertical forces acting at R are the resolved part of Y_8 or $\frac{Y_8}{\sqrt{2}}$, the reaction W of the abutment and the stress in the last vertical, which has already been found $= 3000$ kilos. Hence for equilibrium:

$$\frac{Y_8}{\sqrt{2}} + W - 3000 = 0.$$

Y_8 therefore obtains its greatest negative value when W is a maximum, that is, when the bridge is fully loaded, in which case $W = \frac{48000}{2}$ kilos.; and, therefore,

$$Y_8 \text{ (min.)} = -21000 \times \sqrt{2} = -29700 \text{ kilos.}$$

The results obtained are shown in Fig. 57.

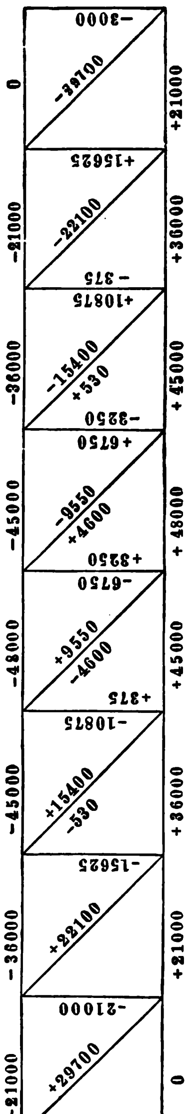


Fig. 57.

§ 11.—Derived Forms.

If the above equations be examined, to ascertain what positions of the live load produce the greatest stress in the diagonal braces, it will be found that the law already found for parabolic girders (p. 26) holds good, or the stress in any diagonal is a maximum or a minimum when the joints on one side only are loaded.

The stresses in a girder in which the diagonals slope upwards from left to right (instead of from right to left) can evidently be obtained by looking at the girder of Fig. 57 from behind.*

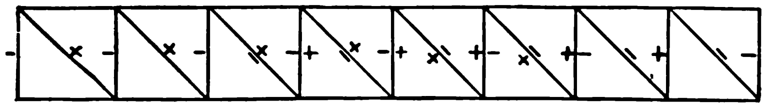
If the diagonals are to be tension-braces, and unable to resist compression, the following alterations will have to be made in the arrangement of the original girder; in the bays where the diagonals are always in compression, they must be changed for diagonals sloping in the opposite direction, and in the bays where the diagonal braces are subject alternately to tension and to compression, two diagonals must be introduced.

Figs. 58 and 59 represent two girders having opposite diagonal systems, and the kind of stress in each brace is denoted

* Holding the page up to the light.—Trans.

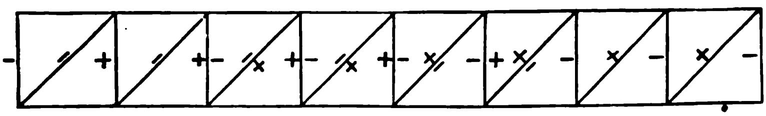
by the sign +, signifying *tension*, and — *compression*. Carrying out the above alterations, Fig. 60 is obtained, in which the diagonals are never in compression, and the greatest numerical value of the tension in them can at once be written down by means of Figs. 58 and 59, taking the values from Fig. 57.

FIG. 58.



A vertical brace can only be in tension when the diagonals meeting it at an unloaded joint are in compression. This can never occur in Fig. 60; and the verticals can, therefore, only be in compression; consequently, only the values of V (min.)

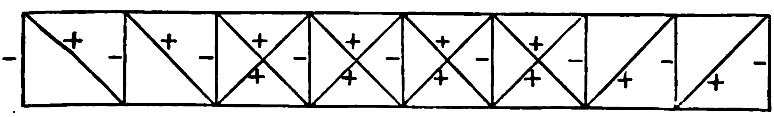
FIG. 59.



need be considered, and for the left-hand side of the girder these values must be taken from Fig. 58, and for the right-hand side from Fig. 59.

The stresses in the horizontal bars X and Z are greatest when

FIG. 60.



the girder is fully loaded, and when this is the case it is easily seen that, in the left half of the girder, the diagonals sloping upwards from right to left will be in tension, and in the right half those sloping from left to right. Evidently, therefore, the stresses in the booms can be obtained from Fig. 58 for the left half, and from Fig. 59 for the right half of the new girder.

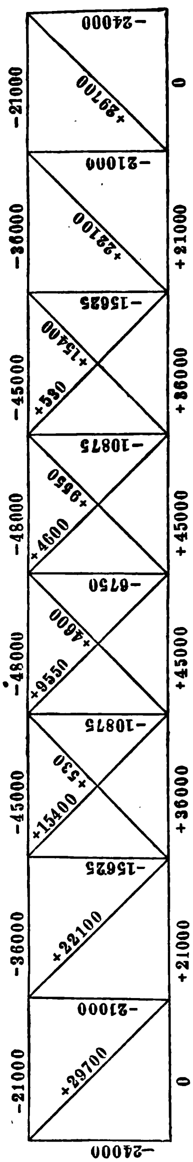


FIG. 61.

Thus, without further calculation, the stresses in a girder of the form shown in Fig. 60 can be obtained. These stresses have been given in Fig. 61.

If the diagonals can only resist compression (as is often the case in wooden structures), the stresses can be obtained by an exactly similar process from Fig. 57. These stresses are shown in Fig. 62.

If the line of railway is carried on the lower apices, instead of on the upper, it can be considered that both the live and the dead loads are applied to the lower joints. The stresses in the horizontal and diagonal bars will not thereby be altered, and the stress in the verticals can be found by the rule of ~~Section~~ 9, namely, that the diagonal and vertical braces meeting at an unloaded joint have equal vertical stresses, but of contrary sign. In this case the unloaded joints are the upper ones, and in Fig. 63 the stress in any vertical can be found by dividing the stress in the diagonal meeting it at the top joint by $\sqrt{2}$, and changing the sign. From Fig. 63 the derived forms shown in Figs. 64 and 65 can be deduced as before.

If the line is carried on the verticals between the booms, the points of attachment can also be considered as the points of application of the live and dead loads. All the upper as well as the lower joints are therefore unloaded, consequently the resolved part vertically of the stress in any diagonal will be the numerical value of the stress in the parts of both the verticals it meets. For instance, in Fig. 66 the diagonal in the third bay is subject to the maximum and minimum stresses

$$+15400 \text{ and } -530 \text{ kilos.}$$

FIG. 62.

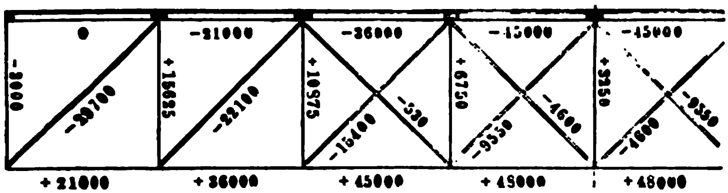


FIG. 64.

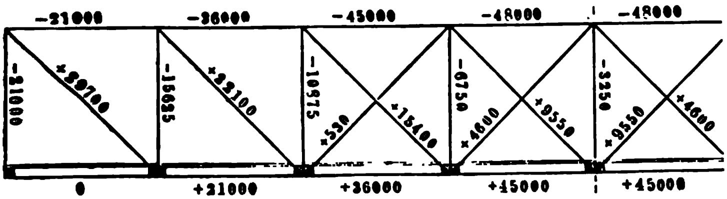


FIG. 65.

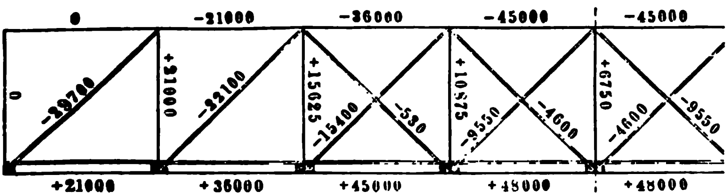


FIG. 67.

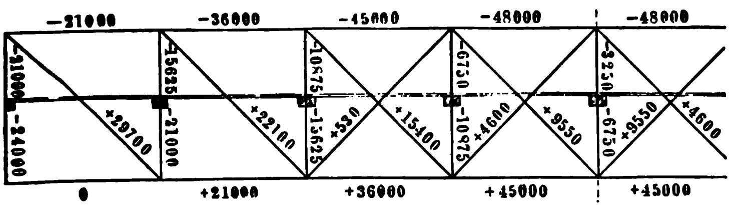
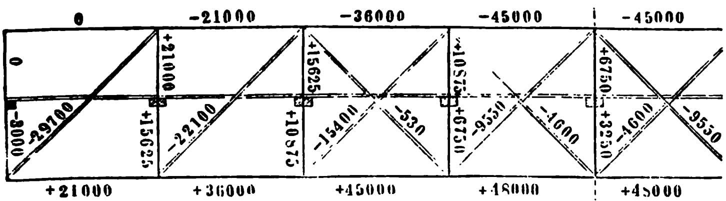


FIG. 68.



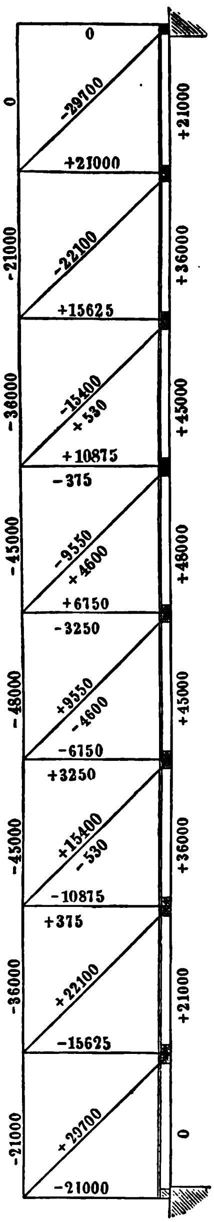


FIG. 63.

These values divided by $\sqrt{2}$ and with changed signs give

$$-10875 \text{ and } +375 \text{ kilos,}$$

and these will be the stresses in the upper part of the vertical to the left and in the lower part of the vertical to the right. The stresses in the verticals in this figure as well as in the girders shown in Figs. 67 and 68 can therefore be obtained without difficulty, using the rule given in § 9. As to the stresses in the horizontal and diagonal bars it is evidently immaterial whether the loads be carried on the top or bottom joints or between them. Lastly in girders with symmetrically arranged diagonals all the stresses can be written down from Figs. 57, 63, and 66, with the exception of the stress in the central vertical. For this reason only the central part of the girder is shown in Figs. 69, 70, 71, 72, 73, and 74, and it is easy to see that the stress in the central vertical will be either ± 6000 kilos. or 0 according as the end which does not meet a diagonal is loaded or not.

§ 12.—REMARKS ON THE DEGREE OF ACCURACY OF THE ASSUMPTIONS MADE WITH REGARD TO THE DISTRIBUTION OF THE LOADS.

Some objections may be raised to the above calculations, for the distribution of the loads on which they are based is not strictly true, and the results to be accurate require a slight correction.

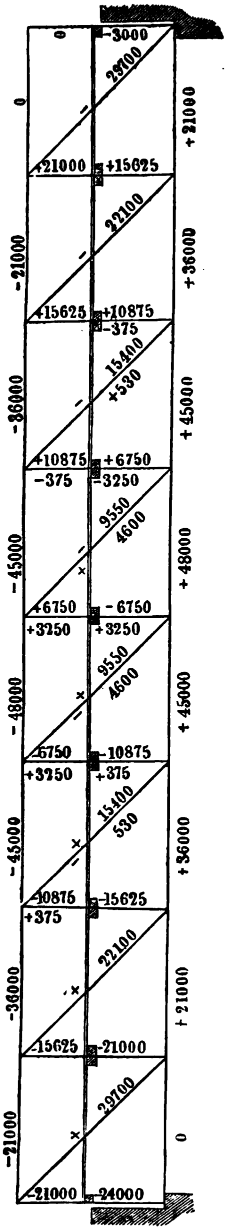


Fig. 66.

Fig. 69.

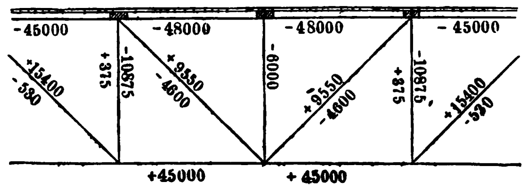


Fig. 70.

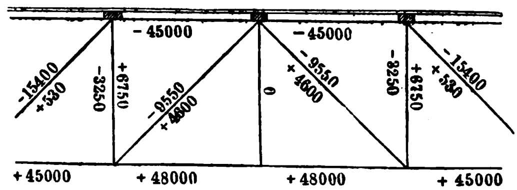


Fig. 71.

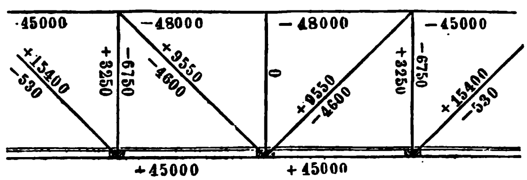


Fig. 72.

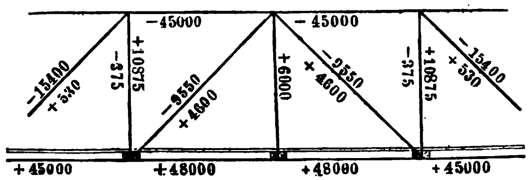
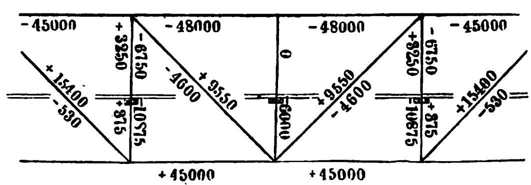
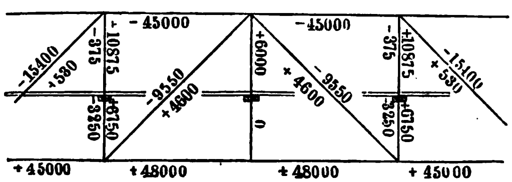


Fig. 73.



In the first place, the weight of the girder itself acts on the upper as well as on the lower joints, and not, as assumed, at the points through which the line of rails passes only. The correction in this case, however, will only apply to the vertical braces; for, as already seen, the stress in the remaining bars is independent of the position of the rails. Taking any of the verticals in Fig. 53 or Fig. 54 and distributing the load on it in due proportion between the top and bottom, it is easily seen that the method of moments could be applied to find the stress in that vertical. But it is better to make the calculation as in § 9 and § 10 (i. e. taking the point of application of the dead load the same as that of the live load), and then, if considered necessary, apply a correction in the following manner: Imagine

FIG. 74.



a secondary vertical placed alongside of the main one, the object of this vertical being to realize the assumption made by transmitting the load on what has been considered the unloaded joint to the loaded joint. This secondary vertical will be a strut when the load has to be transmitted downwards, and a tie when it has to be transmitted upwards. The stress in it will therefore be negative when it is above the line of rails, and positive when it is below. Now if the secondary vertical be considered amalgamated with the principal vertical, it is evident that the actual stress in the latter can be found by adding to the stress already determined the stress in the former.

To make this clearer by an example, let the true distribution of the dead load in Fig. 57 be $\frac{2}{3}$rds of the 1000 kilos. on the top joints and $\frac{1}{3}$rd on the bottom joints, whereas it was considered that the whole of the dead load was applied to the top joints. The secondary vertical has therefore to transmit 333 kilos. from the lower to the upper joints, and is consequently

a tie with a stress of + 333 kilos. This must now be added to the stress in all the verticals. For instance, in vertical V_3

$$V_3\,(\text{max.}) = +\,375 + 333 = 708 \text{ kilos.}$$
$$V_3\,(\text{min.}) = -\,10875 + 333 = -\,10542 \text{ kilos.}$$

In the girder shown in Fig. 63 the line of rails is attached to the bottom joints. Supposing that the true distribution of the dead load is $\frac{1}{3}$rd on the upper joints and $\frac{2}{3}$rds on the lower joints, it is evident that the secondary verticals will be struts transmitting 333 kilos. from the upper to the lower joints, and therefore to obtain the correct stress in the verticals — 333 kilos. must be added to the stresses already found. For instance in vertical V_3

$$V_3\,(\text{max.}) = +\,3250 - 333 = +\,2917 \text{ kilos.}$$
$$V_3\,(\text{min.}) = -\,6750 - 333 = -\,7083 \text{ kilos.}$$

In this case the correction is so small that it might be neglected. But in larger bridges, where the dead load is large in comparison to the live load, and is more equally distributed on the joints, the correction becomes important.

There is a second correction to be made, in connection with the distribution of the moving load. It will be remembered that it was assumed that each bay was bridged over by secondary girders,* so as to convey the dead and live loads on the line of rails to the adjacent joints. It is evident that it is only when a bay is fully loaded that the reaction at each end of the secondary girder can be equal to half the load on one bay. Now the stresses in the diagonal and vertical braces were calculated on the supposition that all the joints on one side of the brace were fully loaded, and all those on the other side free of the moving load. With a uniformly distributed moving load this obviously cannot occur.

Yet, when it is considered that in reality the moving load is not uniform and continuous, but that, on the contrary, the load is concentrated at the points of contact of the wheels with

* These secondary girders are supposed to be discontinuous.—TRANS.

the rails, and that in the case when the distance between the wheels is equal to the distance between the joints, the above assumption is strictly true, it would appear that the error is insignificant, unless indeed the number of bays is small. At any rate the error affects only the diagonals and verticals, and is on the safe side.

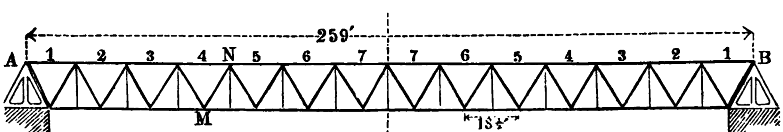
Both these sources of error disappear, the latter when the number of bays is very great, and the first when there are no verticals, in which case the calculations differ slightly from the preceding ones. To illustrate this latter point, the following example has been chosen.

§ 13.—Braced Girder with Equilateral Triangles. (Warren Girder.)

(*Railway Bridge over the Trent near Newark.*)

Each girder (Fig. 75) is composed of 27 bays of equilateral triangles having their apices alternately above and below. The line of rails is on a level with the bottom boom, one half of the load is carried directly on the lower joints, and the other half is transmitted by means of vertical ties to the upper joints. Thus one half of the dead as well as of the live load acts on the lower joints, and the remaining half on the upper joints. The

Fig. 75.



whole girder is supported at A and B by bolts carried on cast-iron frames which rest on the piers. The distance apart of these points of support is 259 feet, and therefore the length of the side of one of the triangles is $\frac{259}{14} = 18 \cdot 5$ feet. The depth of the girder is $\frac{18 \cdot 5}{2} \times \tan. 60° = 9 \cdot 25 \times 1 \cdot 73$ feet. Taking $9 \cdot 25$

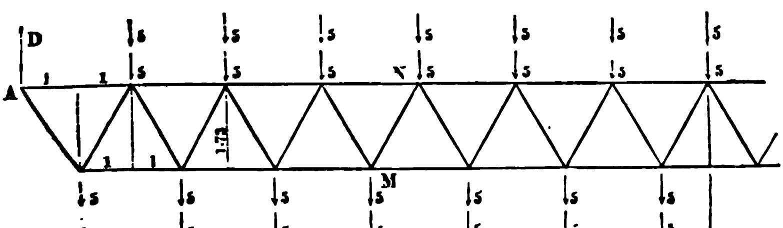
feet as the unit of length, the side of the triangles will be represented by 2, the height of the girder by 1·73, and its length by 28.

The weight of the whole bridge is 589 tons, and since there are four girders, the dead load on each girder is $\frac{589}{4} = 147\frac{1}{4}$ tons. Taking the moving load on each line at 1 ton per foot run, it will amount to $\frac{2 \times 259}{4} = 129 \cdot 5$ tons on each girder. The total load on one girder is therefore

$$147 \cdot 25 + 129 \cdot 5 = 276 \cdot 75 \text{ tons.}$$

Thus the load on each joint is $\frac{276 \cdot 75}{28}$ tons, or in round numbers 10 tons. In the following calculations the live as well as the dead load has been taken for simplicity at 5 tons on each joint, although the proportion of the dead to the live load is as 147·25 : 129·5; this will make no difference in the stresses in the booms, and the greatest stresses in the diagonals will be slightly increased. Besides, the live load cannot be considered as accurately equal to 1 ton per foot run; often it is taken higher.

FIG. 76.



The only object of the vertical bars is to transmit part of the load to the upper apices; they do not form an integral part of the truss, and can therefore be omitted in the calculations, the upper joints being considered loaded instead. The distribution of the load is therefore as shown in Fig. 76.

Calculation of the Stresses X and Z in the Upper and Lower Booms.

Cutting off the part shown in Fig. 77 by the section line $\alpha\beta$, and taking moments first about the point M and then about the point N, the following equations are obtained, denoting by D the reaction at the abutment A:

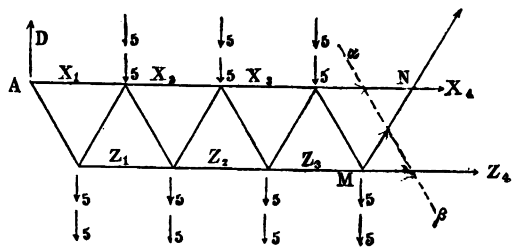
$$0 = X_4 \times 1{\cdot}73 + D \times 7 - 5(1+2+3+4+5+6) \\ - 5(1+2+3+4+5+6)$$

$$0 = -Z_4 \times 1{\cdot}73 + D \times 8 - 5(1+2+3+4+5+6+7) \\ - 5(1+2+3+4+5+6+7).$$

Substituting for D its value

$$D = 5\left(\tfrac{1}{28} + \tfrac{2}{28} + \ldots + \tfrac{27}{28}\right) + 5\left(\tfrac{1}{28} + \tfrac{2}{28} + \ldots + \tfrac{27}{28}\right),$$

FIG. 77.



and arranging the equations so that the effect of each load may be seen (according to the previous rule).

$$0 = X_4 \times 1{\cdot}73 \\ + 5\left\{\left(\tfrac{1}{28} + \ldots + \tfrac{21}{28}\right)7 + \left(\tfrac{22}{28}.7 - 1\right) + \left(\tfrac{23}{28}.7 - 2\right) + \ldots + \left(\tfrac{27}{28}.7 - 6\right)\right\} \\ + 5\left\{\left(\tfrac{1}{28} + \ldots + \tfrac{21}{28}\right)7 + \left(\tfrac{22}{28}.7 - 1\right) + \left(\tfrac{23}{28}.7 - 2\right) + \ldots + \left(\tfrac{27}{28}.7 - 6\right)\right\}.$$

$$0 = -Z_4 \times 1{\cdot}73 \\ + 5\left\{\left(\tfrac{1}{28} + \ldots + \tfrac{20}{28}\right)8 + \left(\tfrac{21}{28}.8 - 1\right) + \left(\tfrac{22}{28}.8 - 2\right) + \ldots + \left(\tfrac{27}{28}.8 - 7\right)\right\} \\ + 5\left\{\left(\tfrac{1}{28} + \ldots + \tfrac{20}{28}\right)8 + \left(\tfrac{21}{28}.8 - 1\right) + \left(\tfrac{22}{28}.8 - 2\right) + \ldots + \left(\tfrac{27}{28}.8 - 7\right)\right\}.$$

In these equations all the members containing the live load have positive signs, and the omission of any of them would consequently diminish the numerical value of the stress.

Having thus shown that the booms obtain their greatest stress when the bridge is fully loaded (and this, it may be added, is true in the case of all lattice girders), the calculations

can be simplified by putting for D its value when the moving load covers the bridge, namely,

$$D = 10\left(\tfrac{1}{28} + \tfrac{2}{28} + \ldots + \tfrac{27}{28}\right) = 135 \text{ tons},$$

and combining the members containing the live and dead loads. The above equations then become

$$0 = X_4 \times 1 \cdot 73 + 135 \times 7 - 10(1 + 2 + \ldots + 6)$$
$$0 = -Z_4 \times 1 \cdot 73 + 135 \times 8 - 10(1 + 2 + \ldots + 7),$$

whence

$$X_4 \text{ (min.)} = -425 \text{ tons},$$
$$Z_4 \text{ (max.)} = +462 \text{ tons}.$$

The following equations, for the remaining parts of the booms, are obtained in an exactly similar manner:

$$0 = X_1 \times 1 \cdot 73 + 135 \times 1$$
$$X_1 \text{ (min.)} = -78 \text{ tons}.$$

$$0 = -Z_1 \times 1 \cdot 73 + 135 \times 2 - 10 \times 1$$
$$Z_1 \text{ (max.)} = +150 \text{ tons}.$$

$$0 = X_2 \times 1 \cdot 73 + 135 \times 3 - 10(1 + 2)$$
$$X_2 \text{ (min.)} = -216 \text{ tons}.$$

$$0 = -Z_2 \times 1 \cdot 73 + 135 \times 4 - 10(1 + 2 + 3)$$
$$Z_2 \text{ (max.)} = +277 \text{ tons}.$$

$$0 = X_3 \times 1 \cdot 73 + 135 \times 5 - 10(1 + 2 + 3 + 4)$$
$$X_3 \text{ (min.)} = -338 \text{ tons}.$$

$$0 = -Z_3 \times 1 \cdot 73 + 135 \times 6 - 10(1 + 2 + 3 + 4 + 5)$$
$$Z_3 \text{ (max.)} = +381 \text{ tons}.$$

$$0 = X_5 \times 1 \cdot 73 + 135 \times 9 - 10(1 + 2 + \ldots + 8)$$
$$X_5 \text{ (min.)} = -494 \text{ tons}.$$

$$0 = -Z_5 \times 1 \cdot 73 + 135 \times 10 - 10(1 + 2 + \ldots + 9)$$
$$Z_5 \text{ (max.)} = +520 \text{ tons}.$$

$$0 = X_6 \times 1.73 + 135 \times 11 - 10(1 + 2 + \ldots + 10)$$
$$X_6 \text{ (min.)} = -540 \text{ tons}.$$

$$0 = -Z_6 \times 1 \cdot 73 + 135 \times 12 - 10(1 + 2 + \ldots + 11)$$
$$Z_6 \text{ (max.)} = +555 \text{ tons}.$$

$$0 = X_7 \times 1 \cdot 73 + 135 \times 13 - 10(1 + 2 + \ldots + 12)$$
$$X_7 \text{ (min.)} = -564 \text{ tons}.$$

$$0 = -Z_7 \times 1 \cdot 73 + 135 \times 14 - 10(1 + 2 + \ldots + 13),$$
$$Z_7 \text{ (max.)} = +566 \text{ tons}.$$

Calculation of the Stresses Y and U in the Braces.

Since the braces make an angle of 60° with the horizontal, the resolved part vertically of the stresses in them are

$$Y \,.\, \sin 60° \quad \text{and} \quad U \,.\, \sin 60°$$

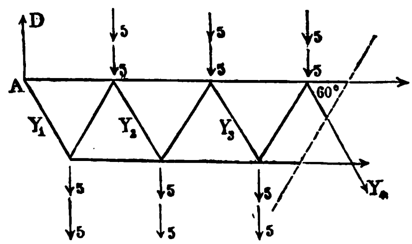
or

$$Y \times 0{\cdot}866 \quad \text{and} \quad U \times 0{\cdot}866.$$

Resolving the forces acting on the parts of the girder shown in Figs. 78 and 79 vertically.

$$0 = Y_4 \times 0{\cdot}866 - D + 5 \times 6 + 5 \times 6$$
$$0 = - U_4 \times 0{\cdot}866 - D + 5 \times 7 + 5 \times 7.$$

FIG. 78.



Substituting for D its value

$$D = 5\left(\tfrac{1}{28} + \tfrac{2}{28} + \ldots + \tfrac{27}{28}\right) + 5\left(\tfrac{1}{28} + \tfrac{2}{28} + \ldots + \tfrac{27}{28}\right),$$

and arranging the equations according to the previous rule

$$\begin{aligned} 0 = {} & Y_4 \times 0{\cdot}866 \\ & - 5\left\{\tfrac{1}{28} + \tfrac{2}{28} + \ldots + \tfrac{21}{28} - \left(1 - \tfrac{22}{28}\right) - \left(1 - \tfrac{23}{28}\right) - \ldots - \left(1 - \tfrac{27}{28}\right)\right\} \\ & - 5\left(\tfrac{1}{28} + \ldots + \tfrac{21}{28}\right) + 5\left\{\left(1 - \tfrac{22}{28}\right) + \left(1 - \tfrac{23}{28}\right) + \ldots + \left(1 - \tfrac{27}{28}\right)\right\} \end{aligned}$$

$$\begin{aligned} 0 = {} & - U_4 \times 0{\cdot}866 \\ & - 5\left\{\tfrac{1}{28} + \ldots + \tfrac{20}{28} - \left(1 - \tfrac{21}{28}\right) - \left(1 - \tfrac{22}{28}\right) - \ldots - \left(1 - \tfrac{27}{28}\right)\right\} \\ & - 5\left(\tfrac{1}{28} + \ldots + \tfrac{20}{28}\right) + 5\left\{\left(1 - \tfrac{21}{28}\right) + \left(1 - \tfrac{22}{28}\right) + \ldots + \left(1 - \tfrac{27}{28}\right)\right\}. \end{aligned}$$

In these equations the members containing the moving load do not all possess the same sign, therefore leaving out first the positive and then the negative members,

$$\begin{aligned} 0 = {} & Y_4 \times 0{\cdot}866 - 5\left(\tfrac{1}{28} + \ldots + \tfrac{21}{28} - \tfrac{6}{28} - \tfrac{5}{28} - \ldots - \tfrac{1}{28}\right) \\ & - 5\left(\tfrac{1}{28} + \ldots + \tfrac{21}{28}\right) \end{aligned}$$

$$X_4 \text{ (max.)} = + 91 \text{ tons.}$$

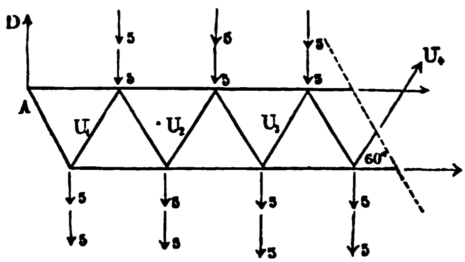
$$0 = Y_4 \times 0{\cdot}866 - 5\left(\tfrac{1}{28} + \ldots + \tfrac{21}{28} - \tfrac{6}{28} - \tfrac{5}{28} - \ldots - \tfrac{1}{28}\right)$$
$$+ 5\left(\tfrac{6}{28} + \tfrac{5}{28} + \ldots + \tfrac{1}{28}\right)$$
$$X_4\ (\text{min.}) = +\ 39 \text{ tons.}$$

$$0 = -\,U_4 \times 0{\cdot}866 - 5\left(\tfrac{1}{28} + \ldots + \tfrac{20}{28} - \tfrac{7}{28} - \tfrac{6}{28} - \ldots - \tfrac{1}{28}\right)$$
$$+ 5\left(\tfrac{7}{28} + \tfrac{6}{28} + \ldots + \tfrac{1}{28}\right)$$
$$U_4\ (\text{max.}) = -\,82 \text{ tons.}$$

$$0 = -\,U_4 \times 0{\cdot}866 - 5\left(\tfrac{1}{28} + \ldots + \tfrac{20}{28} - \tfrac{7}{28} - \tfrac{6}{28} - \ldots - \tfrac{1}{28}\right)$$
$$- 5\left(\tfrac{1}{28} + \tfrac{2}{28} + \ldots + \tfrac{20}{28}\right)$$
$$U_4\ (\text{min.}) = -\,81 \text{ tons.}$$

It appears that only Y_4 (max.) and U_4 (min.) need be taken into consideration, and therefore the calculations for Y_4 (min.) and U_4 (max.) could have been spared. But it is advisable always to calculate both values, for sometimes the stresses are of different signs, in which case both must be retained.

Fig. 79.



The equations and the stresses in the remaining diagonals are found similarly as shown below:

$$0 = Y_1 \times 0{\cdot}866 - 5\left(\tfrac{1}{28} + \ldots + \tfrac{27}{28}\right) - 5\left(\tfrac{1}{28} + \ldots + \tfrac{27}{28}\right)$$
$$Y_1 \begin{cases} (\text{max.}) = +\ 156 \text{ tons} \\ (\text{min.}) = +\ \ 78 \text{ tons} \end{cases}$$

$$0 = -\,U_1 \times 0{\cdot}866 - 5\left(\tfrac{1}{28} + \ldots + \tfrac{26}{28} - \tfrac{1}{28}\right) - 5\left(\tfrac{1}{28} + \ldots + \tfrac{26}{28}\right)$$
$$+ 5 \times \tfrac{1}{28}$$
$$U_1 \begin{cases} (\text{max.}) = -\ \ 72 \text{ tons} \\ (\text{min.}) = -\ 144 \text{ tons} \end{cases}$$

$$0 = Y_2 \times 0{\cdot}866 - 5\left(\tfrac{1}{28} + \ldots + \tfrac{25}{28} - \tfrac{2}{28} - \tfrac{1}{28}\right)$$
$$- 5\left(\tfrac{1}{28} + \ldots + \tfrac{25}{28}\right) + 5\left(\tfrac{2}{28} + \tfrac{1}{28}\right)$$
$$Y_2 \begin{cases} (\text{max.}) = +\ 133 \text{ tons} \\ (\text{min.}) = +\ \ 66 \text{ tons} \end{cases}$$

$$\begin{aligned}0 = -\,U_2 \times 0{\cdot}866 &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{24}{28} - \tfrac{3}{28} - \tfrac{2}{28} - \tfrac{1}{28}\right)\\ &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{24}{28}\right) + 5\left(\tfrac{3}{28} + \tfrac{2}{28} + \tfrac{1}{28}\right)\end{aligned}$$

$$U_2 \begin{cases} (\text{max.}) = - \ \ 59 \text{ tons} \\ (\text{min.}) = - 122 \text{ tons} \end{cases}$$

$$\begin{aligned}0 = Y_3 \times 0{\cdot}866 &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{23}{28} - \tfrac{4}{28} - \ldots - \tfrac{1}{28}\right)\\ &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{23}{28}\right) + 5\left(\tfrac{4}{28} + \ldots + \tfrac{1}{28}\right)\end{aligned}$$

$$Y_3 \begin{cases} (\text{max.}) = + 112 \text{ tons} \\ (\text{min.}) = + \ \ 53 \text{ tons} \end{cases}$$

$$\begin{aligned}0 = -\,U_3 \times 0{\cdot}866 &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{22}{28} - \tfrac{5}{28} - \ldots - \tfrac{1}{28}\right)\\ &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{22}{28}\right) + 5\left(\tfrac{5}{28} + \ldots + \tfrac{1}{28}\right)\end{aligned}$$

$$U_3 \begin{cases} (\text{max.}) = - \ \ 46 \text{ tons} \\ (\text{min.}) = - 101 \text{ tons} \end{cases}$$

$$\begin{aligned}0 = Y_5 \times 0{\cdot}866 &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{19}{28} - \tfrac{8}{28} - \ldots - \tfrac{1}{28}\right)\\ &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{19}{28}\right) + 5\left(\tfrac{8}{28} + \ldots + \tfrac{1}{28}\right)\end{aligned}$$

$$Y_5 \begin{cases} (\text{max.}) = + 71 \text{ tons} \\ (\text{min.}) = + 24 \text{ tons} \end{cases}$$

$$\begin{aligned}0 = -\,U_5 \times 0{\cdot}866 &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{18}{28} - \tfrac{9}{28} - \ldots - \tfrac{1}{28}\right)\\ &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{18}{28}\right) + 5\left(\tfrac{9}{28} + \ldots + \tfrac{1}{28}\right)\end{aligned}$$

$$U_5 \begin{cases} (\text{max.}) = - 17 \text{ tons} \\ (\text{min.}) = - 61 \text{ tons} \end{cases}$$

$$\begin{aligned}0 = Y_6 \times 0{\cdot}866 &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{17}{28} - \tfrac{10}{28} - \ldots - \tfrac{1}{28}\right)\\ &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{17}{28}\right) + 5\left(\tfrac{10}{28} + \ldots + \tfrac{1}{28}\right)\end{aligned}$$

$$Y_6 \begin{cases} (\text{max.}) = + 52 \text{ tons} \\ (\text{min.}) = + \ \ 9 \text{ tons} \end{cases}$$

$$\begin{aligned}0 = -\,U_6 \times 0{\cdot}866 &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{16}{28} - \tfrac{11}{28} - \ldots - \tfrac{1}{28}\right)\\ &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{16}{28}\right) + 5\left(\tfrac{11}{28} + \ldots + \tfrac{1}{28}\right)\end{aligned}$$

$$U_6 \begin{cases} (\text{max.}) = - 0{\cdot}8 \text{ tons} \\ (\text{min.}) = - \ \ 42 \text{ tons} \end{cases}$$

$$\begin{aligned}0 = X_7 \times 0{\cdot}866 &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{15}{28} - \tfrac{12}{28} - \ldots - \tfrac{1}{28}\right)\\ &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{15}{28}\right) + 5\left(\tfrac{12}{28} + \ldots + \tfrac{1}{28}\right)\end{aligned}$$

$$X_7 \begin{cases} (\text{max.}) = + \ \ 34 \text{ tons} \\ (\text{min.}) = - 7{\cdot}4 \text{ tons} \end{cases}$$

$$\begin{aligned}0 = -\,U_7 \times 0{\cdot}866 &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{14}{28} - \tfrac{13}{28} - \ldots - \tfrac{1}{28}\right)\\ &- 5\left(\tfrac{1}{28} + \ldots + \tfrac{14}{28}\right) + 5\left(\tfrac{13}{28} + \ldots + \tfrac{1}{28}\right)\end{aligned}$$

$$U_7 \begin{cases} (\text{max.}) = + 16 \text{ tons} \\ (\text{min.}) = - 25 \text{ tons.} \end{cases}$$

Since the girder is symmetrical with respect to the central line, the stresses in the corresponding braces in the other half will be exactly the same, and need not therefore be calculated.

The verticals have to sustain a part of the dead load as well as the 5 tons moving load. Half the weight of the line of railway, which forms part of the dead load, is supported by the lower joints, and the other half is transmitted by the verticals to the upper joints. This weight is 24·75 tons,

and since there are fourteen bays, each vertical will have $\frac{24 \cdot 75}{2 \times 14} = 0 \cdot 88$ ton to carry. Each vertical is therefore subject to a tension of $+ 5 \cdot 88$ tons.

The results of the above calculations are collected together in Fig. 80.

If all the signs of the stresses be changed, those in a similar girder turned upside down (Fig. 81) are obtained.

[NOTE.—Since the loads are equally distributed on the top and bottom joints, it is immaterial, so far as regards the booms and diagonals, whether the line be placed on a level with the bottom or the top boom. In the first case the verticals will be ties, and in the second they will be struts.]

§ 14.

In order that the relation between braced girders with a single triangulation, and those having two or more (or Trellis and Lattice girders as they are sometimes called), may be clearly shown, the girders in the following examples will have a span of 16 metres and a total load of 48,000 kilos., so that they may be compared with the girder of § 10.

FIG. 80.

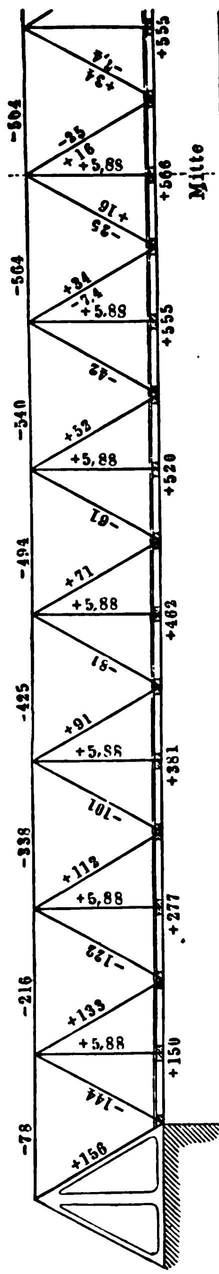
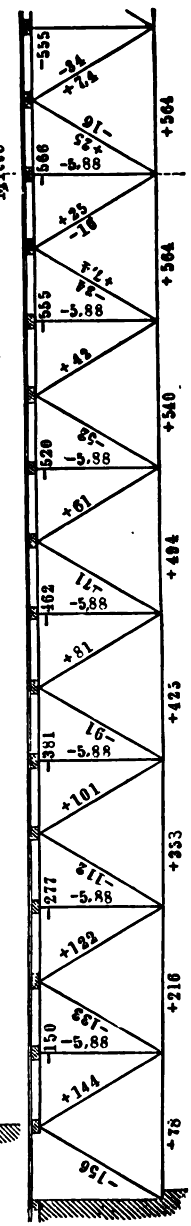


FIG. 81.

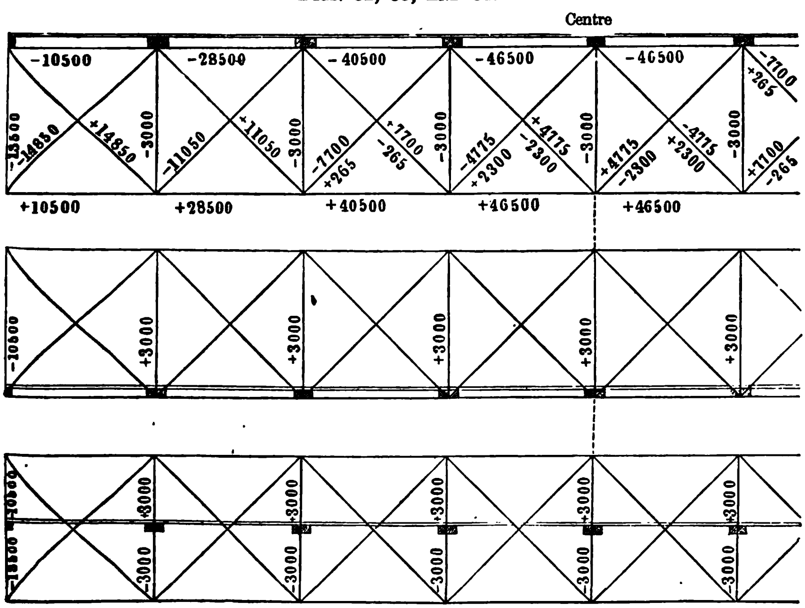


If the load on the girder shown in Fig. 57 were only half what it was assumed to be, the stresses would be exactly a half of those given in the figure. Taking two such girders with halved stresses, one in which the diagonals are inclined upwards from right to left as in Fig. 57, and the other from left to right as in Fig. 63, and placing them exactly one behind the other, so that all the corresponding bars, with the exception of the diagonals, cover each other, a girder is obtained the crossed diagonals of which are capable of taking up either tension or compression. Wherever two bars coincide the stresses in them are to be added, and the stresses thus obtained are those produced in the derived girder by the total original load, one-half of which acts on the upper apices, and the other half on the lower apices. In each of the verticals, except those over the abutments, the stress vanishes, for the maximum stress in one girder is added to the minimum stress in the other. To produce the above loading it is necessary, when the whole load is applied at the level of the upper boom, to introduce verticals to transmit half of it to the lower joints, in which case these verticals will be in compression; or if the line of railway is attached to the lower boom, vertical ties must be used to convey half the load to the top joints; and lastly, if the line is placed between the two booms a vertical will be required, the lower half of which will be in compression, and the upper half in tension. The verticals in compression will have a stress of — 3000 kilos., and those in tension of + 3000 kilos. In this manner the stresses given in Figs. 82, 83, 84 have been obtained. (So as not to overload the diagrams with figures, the stresses in the booms and diagonals have been omitted from Figs. 83 and 84. They are the same as those in Fig. 82.)

Again, if two girders of the design shown in Fig. 57, with halved stresses, be placed so that one overlaps the other by half a bay, and if the stresses in the parts of the booms where they overlap be added, the stresses in a braced girder of the form shown in Fig. 85 will be obtained, but they will only be true if the girder be supported as indicated in the figure. If the line of railway is placed on the lower boom or between the two booms, Figs. 63 and 66 can be employed in a similar manner

to form the girders given in Figs. 86 and 87. (The stresses in the booms and diagonals have been omitted, being the same as those in Fig. 85.) If the diagonals in such a girder be so constructed that they can only resist tension, the form of the girder and the stresses in it will be as shown in Figs. 88, 89, 90. (The stresses in the horizontal and diagonal bars in Figs. 89 and 90 are the same as those in Fig. 88.)

FIGS. 82, 83, AND 84.



The girders in the last six figures have a length of 17 metres (instead of 16 metres), and the stresses given are only true if there are two points supports at each abutment.*

If, however, the original span and method of supporting the girders is to be retained, the design shown in Fig. 91, made up of the two simple systems of Figs. 92 and 93, can be employed. The stresses given have been calculated on the supposition that the live as well as the dead load is applied to the upper

* The girders shown in Figs. 85 to 90 are not of much practical use, for although their length is 17 metres, the clear span is only 16 metres.—TRANS.

Fig. 85.

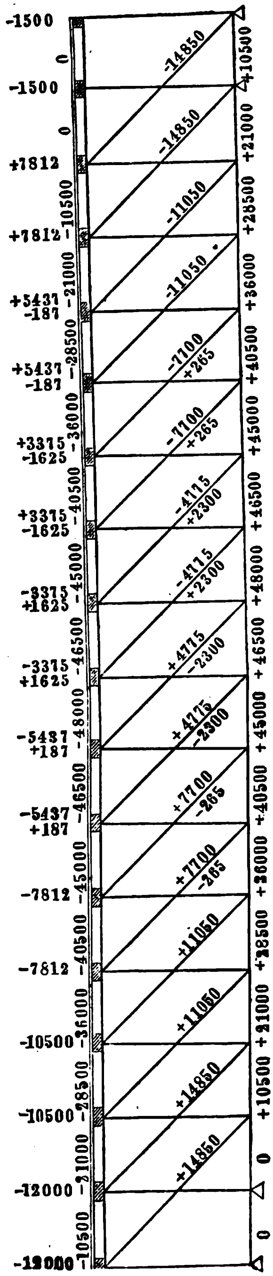


Fig. 86.

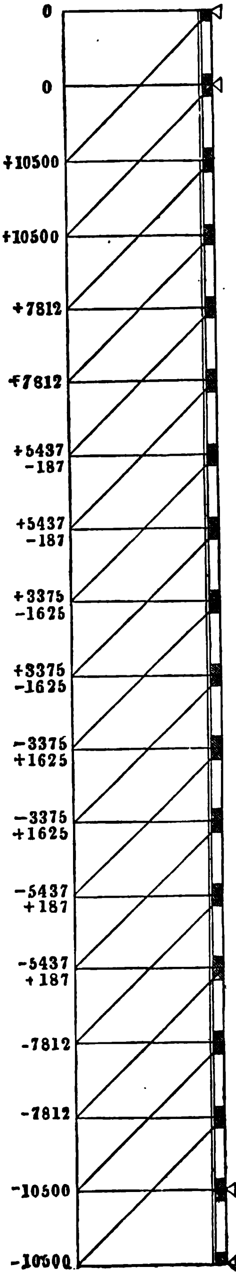
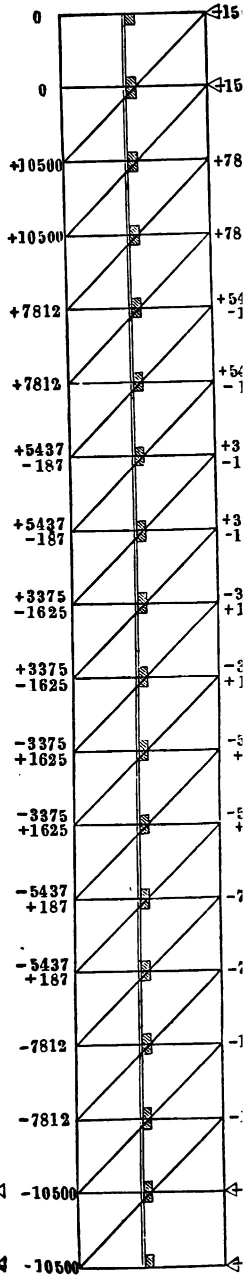


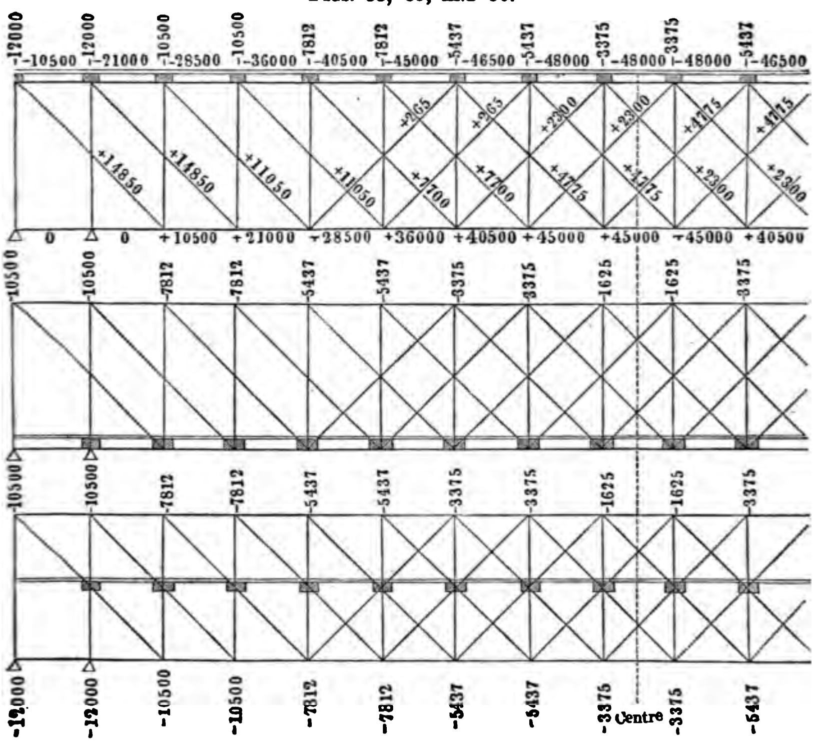
Fig. 87.



extremity of the verticals. The stresses in Fig. 92 are therefore obtained by dividing those in Fig. 57 by two. The stresses in Fig. 93 must, however, be calculated anew, taking the dead load at 4000 kilos. and the moving load at 20,000 kilos.

These calculations are exactly similar to those given in § 10. It is to be observed that in this case, contrary to all

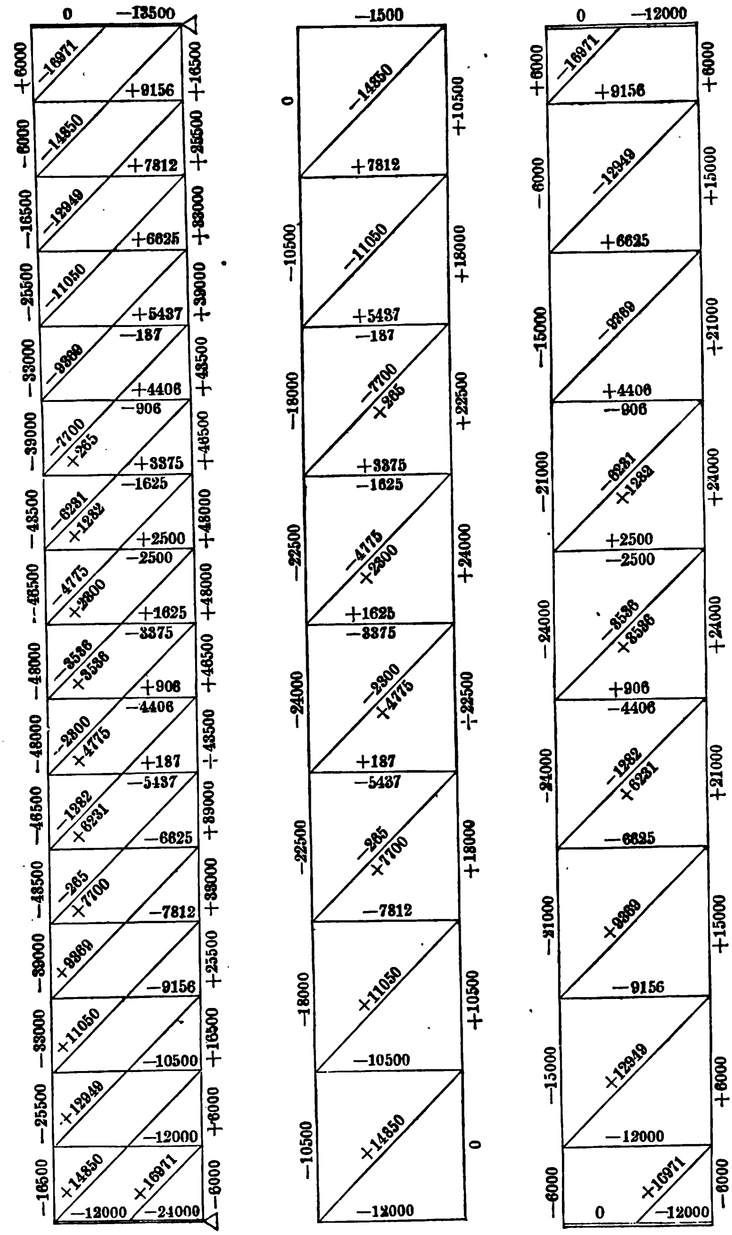
FIGS. 88, 89, AND 90.



the previous examples, the first and last verticals are subject to bending stress (for this reason they have been shown in double lines in the figure). From Fig. 94 the three following equations are obtained for the three bars in the first bay:—

$$0 = X \times 2 + 12000 \times 1 \text{ (turning point P)}$$
$$X = -6000 \text{ kilos.}$$
$$0 = Y \times \frac{1}{\sqrt{2}} - 12000, \qquad Y = 16971 \text{ kilos.}$$
$$0 = -Z \times 2 - 12000 \text{ (turning point 0)}$$
$$Z = -6000 \text{ kilos.}$$

FIG. 91. FIG. 92. FIG. 93.



Both X and Z are negative, these bars are therefore in compression. The first vertical is therefore held in equilibrium by the four forces shown in Fig. 95; thus irrespective of the 12,000 kilos. direct compression in its lower half, it is in the same condition as a beam supported at both ends and loaded in

FIG. 94.

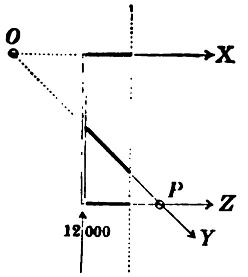
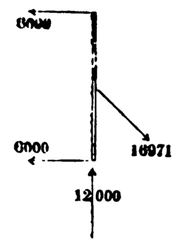


FIG. 95.



the centre with 12,000 kilos. The same figure evidently also represents the stresses in the last vertical.

To avoid these bending stresses the first and last diagonal can be placed as represented in Fig. 96, in which case the equations for the three bars of the first bay will become (Figs. 97 and 98)

$$0 = X \times 2 + 12000 \times 1 \text{ (turning point P)}$$
$$X = -6000 \text{ kilos.}$$
$$0 = Y \times \frac{2}{\sqrt{2^2+1}} - 12000$$
$$Y = +13416 \text{ kilos.}$$
$$0 = -Z \times 2 \text{ (turning point J)}$$
$$Z = 0.$$

The stresses in the bars of the last bay will be similarly altered. This alteration will, however, not affect the stresses in the other bars, and they remain the same as in Fig. 93. Combining the design of Fig. 96 with that of Fig. 92, a girder of the form shown in Fig. 99 is obtained. (Only a few of the stresses are given, for the others coincide with those of Fig. 91.)

Starting with the girders of Figs. 91 and 99, a series of derived forms can be obtained by altering the position of the loads and the nature of the diagonals. Following, for instance, the

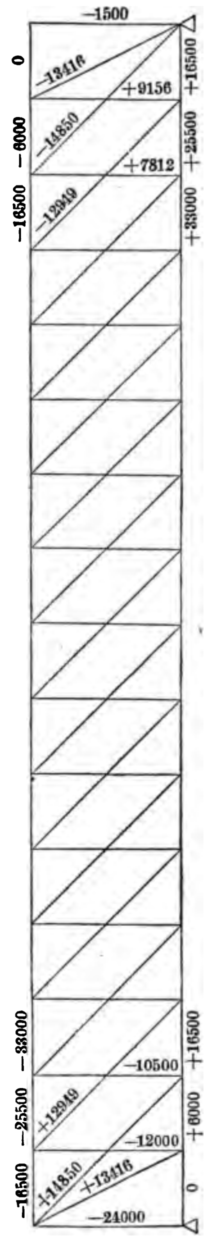


FIG. ~~96~~.99

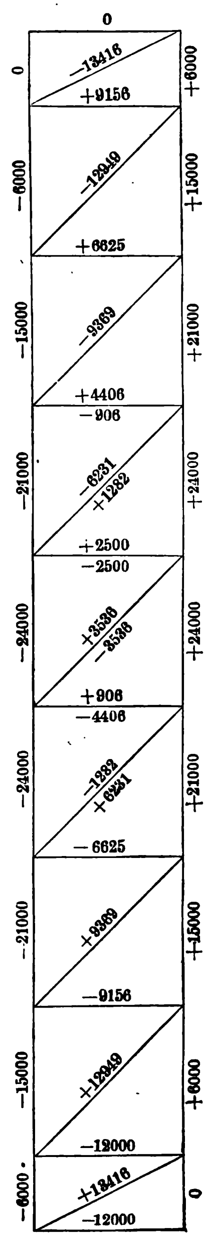


FIG. ~~98~~.96

reasoning of § 11 and assuming that the diagonals can only take up tension, Figs. 100 and 101 are obtained, in which only a half of the girder is shown, for it is symmetrical about the centre, and the stresses in the corresponding bars are equal.

Fig. 102 is obtained by replacing the verticals in Fig. 91 by diagonals inclined to the right at an angle of 45°. This is a trellis girder with four triangulations, and can also be considered as made up of the four girders shown in Figs. 103, 104, 105, and 106.

FIG. 97.

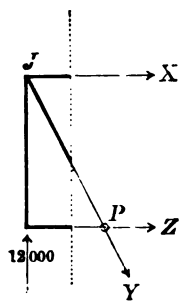
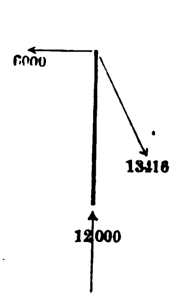
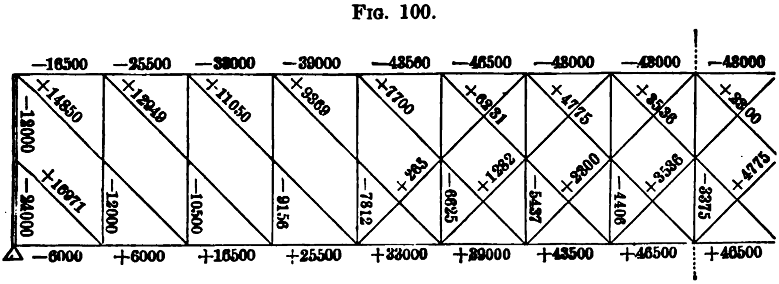


FIG. 98.



The stresses given have been calculated on the supposition that the span of the girders is 16 metres, their height 2 metres; the total dead load $\frac{8000}{4}$ kilos. and the total live load $\frac{40000}{4}$

FIG. 100.



kilos., so that the girder of Fig. 102 may correspond with the former cases.

The dead load, however, according to the more accurate assumption, has been equally distributed between the bottom

Fig. 101.

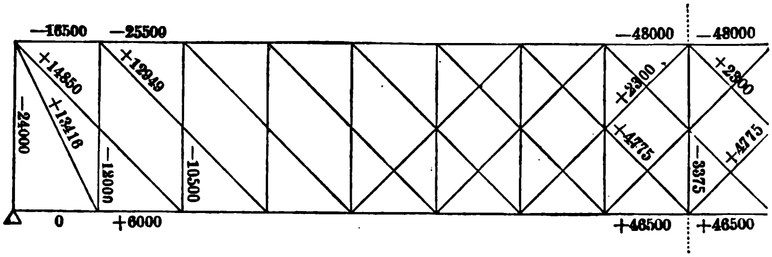


Fig. 102.

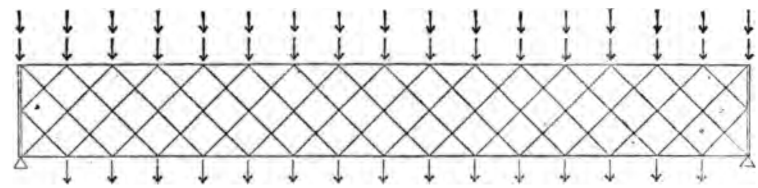


Fig. 103.

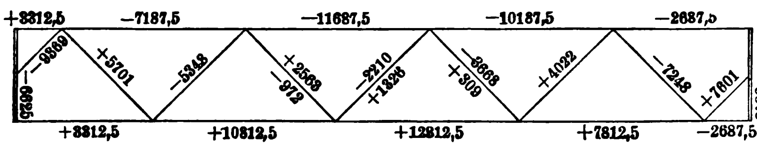


Fig. 104.

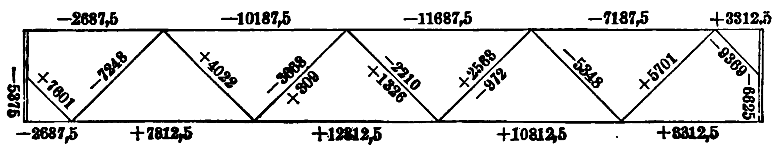


Fig. 105.

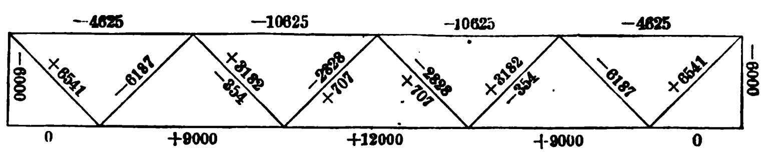
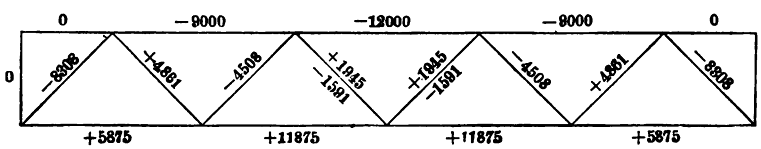


Fig. 106.

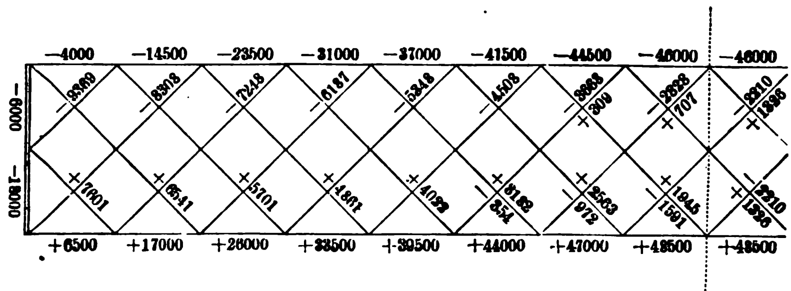


and top joints, but the live load, as in previous cases, is exclusively applied to the upper joints.*

Therefore at each lower joint there is a dead load of 250 kilos. and at each upper joint a dead load of 250 kilos., together with a live load of 2500 kilos. (with the exception of the end joints, which of course have only one-half the load to carry). Fig. 107 is obtained by calculating the stresses in the four single lattice girders with these assumptions and then fuzing them together. The end verticals in Figs. 102, 103, 104, and 107 are represented by double lines to indicate that they are under bending stress.

A comparison of this trellis girder of four triangulations with the one of eight triangulations, shown in Fig. 108, will

FIG. 107.



convince that when the stresses in a trellis girder of a still greater number of triangulations have to be determined it is not necessary to go through all the calculations of all the single systems of which it is composed.

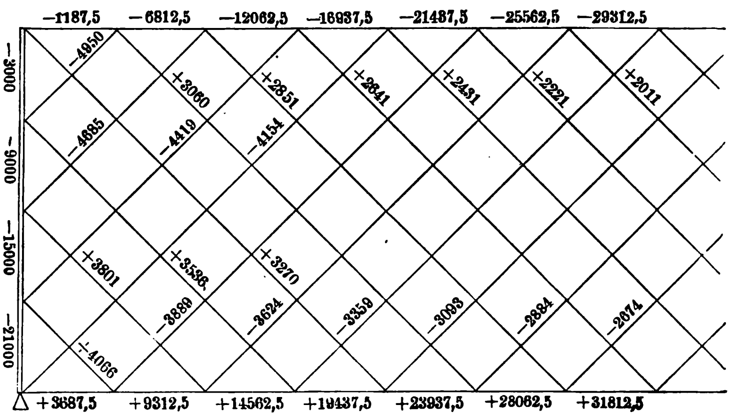
The stresses in the booms increase gradually from the abutments to the centre, and the stresses in the diagonals decrease gradually in the same direction, and these increments and decrements vary according to a law which becomes all the more evident the greater the number of triangulations and the greater the number of stresses actually calculated.

* This assumption is not strictly accurate, for the weight of the line of way, the longitudinal, and the cross-girders is applied to the same joints as the live load, and it is only the weight of the truss itself that can be considered as equally distributed between the bottom and top joints.—TRANS.

Thus, as soon as the calculations have been carried to a certain point, the shorter method of interpolation may be adopted.

The stresses in the diagonals of the girder of Fig. 108 are on an average half those of the corresponding ones of Fig. 107, which agrees with the number of diagonals being double, and the stress in each of the new diagonals is very nearly the arithmetical mean between the two adjacent diagonals. Further, the number representing the stress in any part of the booms of Fig. 107 is almost exactly the arithmetical mean of the two numbers that take its place in Fig. 108. It therefore is not

FIG. 108.



actually necessary to calculate the stresses in each of the eight single lattice girders, forming the girder of Fig. 108, but the stresses already found for lattice girders with two or four triangulations can be used, according to the degree of accuracy required, to determine the required stresses by interpolation.

The points of application and the values of the bending forces acting on the end verticals are given in Figs. 109 and 110. In the first figure the resolved parts vertically and horizontally of the stresses are given instead of the stresses themselves, and in the second figure only the bending forces on the vertical are

shown. The direct compression which also exists in these verticals can be found from Fig. 108.

In Fig. 108 it was assumed that a dead load of 125 kilos. was placed on each lower joint and a dead load of 125 kilos., together with a live load of 1250 kilos. on every upper joint. If instead of this, the dead load on the lower and upper joints had been taken as 1, and the live load as 0, the stresses given in Fig. 111 would have been obtained.

These numbers are what may be called the *stress-numbers* when the bridge has no moving load upon it.

Fig. 109.

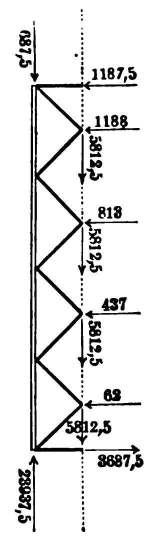
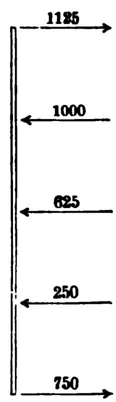


Fig. 110.



These numbers also apply to any similar trellis girder with eight triangulations if the span is eight times the height. If the dead load is p kilos. (or any other unit of weight) on each joint, the stress in kilos. can evidently be obtained by multiplying the stress-numbers by p.

Fig. 112 gives the stress-numbers obtained supposing the dead load to be 0 and the live load on the upper as well as on the lower joints to be 1. To obtain the stress in kilos. of a geometrically similar girder having a live load of m kilos. on the top and bottom joints, the stress-numbers are to be multiplied by m.

The stress-numbers in Fig. 111 are simultaneous, whereas those in Fig. 112 do not occur at the same time, but give the greatest stresses due to partial loading. The functions of the diagonals can therefore be best investigated from Fig. 111, and it will be observed that by multiplying the stress-numbers of this figure by $p + m$, the stresses in the fully-loaded girder are obtained.

Taking two vertical sections through this figure, one at the

FIG. 111.

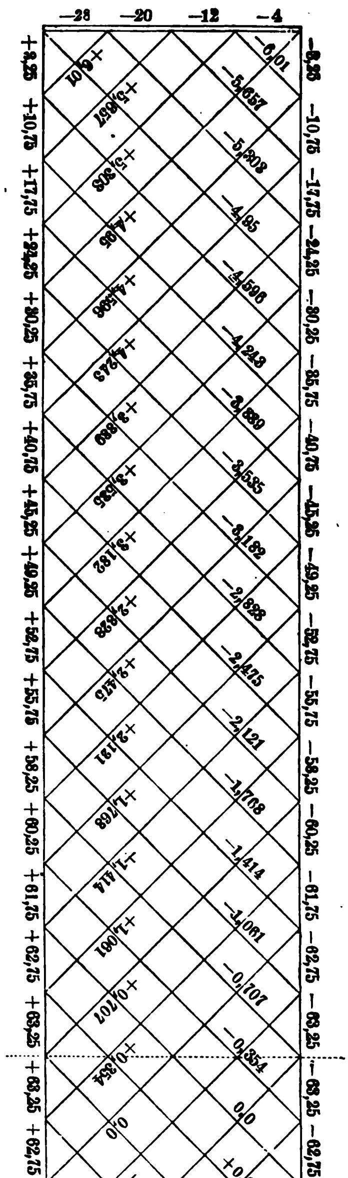
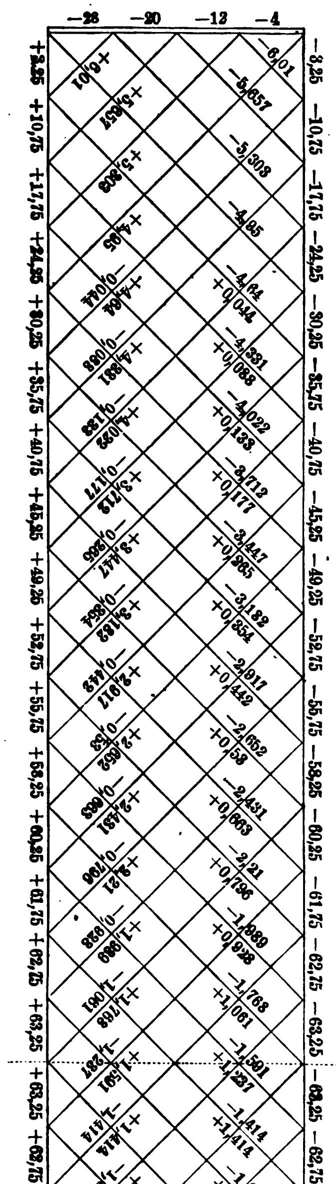
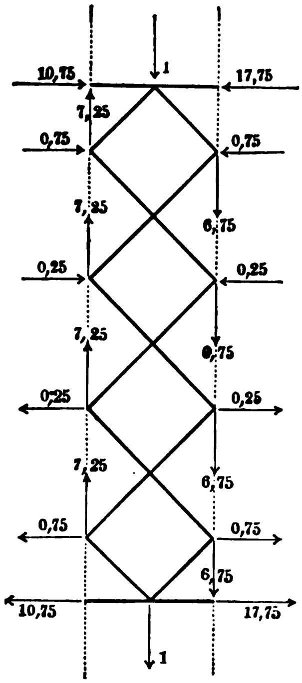


FIG. 112.



first top joint from the left and the other at the second top joint and applying the requisite horizontal and vertical forces to maintain equilibrium, Fig. 113 is obtained. It will be seen that in each section the vertical forces distribute themselves equally between the points of crossing of the diagonals, and that the total difference of the vertical forces is equal to the total load on the bay. It will further be observed that the horizontal forces increase from the centre towards the booms, and are proportional to their distance from the centre. These laws would have been even more clearly expressed with a trellis girder of sixteen triangulations.

FIG. 113.



The greater the number of triangulations, the greater will be the analogy between the functions of the diagonals and those of the solid web of a plate girder.

The stress-numbers in Figs. 111 and 112 can be employed as follows to determine the stress in any other trellis girder with eight triangulations, geometrically similar, and having a dead load of p kilos. and a live load of m kilos. on the upper as well as the lower joints:

Multiply the stress-numbers in Fig. 111 *by* p, *and those in Fig.* 112 *by* m. *The sum of the numbers obtained will give the stress required.*

Or representing by Z_p the stress produced by the dead load, by Z_m the stress produced by the live load, and by Z the total stress:

$$Z = p\,Z_p + m\,Z_m.$$

As an example take a trellis girder of 64 metres span and 8 metres depth, the dead load being p = 1500 kilos. and the

live load m = 2000 kilos. on each top and bottom joint; then for the sixteenth brace inclined upwards from right to left, the above equation becomes

$$Z = 1500 \times 0{\cdot}707 + 2000 \times 1{\cdot}768 = +\ 4596 \text{ kilos.}$$

as the maximum stress or greatest tension, and

$$Z = 1500 \times 0{\cdot}707 - 2000 \times 1{\cdot}061 = -\ 1061 \text{ kilos.}$$

as the minimum stress or greatest compression.

In the same manner the following stresses are obtained for the remaining diagonals inclined upwards from right to left, beginning at the left end of the girder:

$$\begin{array}{llll}
+\ 21036, & +\ 19799, & +\ 18562, & +\ 17324, \\
+\ 16175, & +\ 15027, & +\ 13877, & +\ 12727, \\
+\ 11667, & +\ 10606, & +\ 9546, & +\ 8485, \\
+\ 7513, & +\ 6540, & \begin{cases} +\ 5569, \\ -\ 265, \end{cases} & \begin{cases} +\ 4596, \\ -\ 1061, \end{cases} \\
\begin{cases} +\ 3713, \\ -\ 1944, \end{cases} & \begin{cases} +\ 2828, \\ -\ 2828, \end{cases} & &
\end{array}$$

and these stresses multiplied by − 1 will give the stresses in the diagonals inclined upwards from left to right.

Similarly it is found that the different parts of the booms are subject to the following stresses, commencing at the left end of the girder:

11375, 37625, 62125, 84875, 105875,
135125, 142625, 158375, 172375,
184625, 195125, 203875, 210875,
216125, 219625, 221375.

These numbers taken with the positive sign give the stresses in kilos. in the lower boom and with the negative sign those in the upper boom.

Again if p = 125 kilos. and m = 625 kilos. in the girder of 16^{m} span, the stresses in the diagonals inclined upwards from right to left, beginning at the left end of the girder, are

$$\begin{array}{llll}
+\ 4508, & +\ 4243, & +\ 3977, & +\ 3712, \\
+\ 3475, & +\ 3237, & +\ 3000, & +\ 2762, \\
+\ 2552, & +\ 2342, & +\ 2132, & \begin{cases} +\ 1922, \\ -\ 66, \end{cases} \\
\begin{cases} +\ 1740, \\ -\ 193, \end{cases} & \begin{cases} +\ 1558, \\ -\ 320, \end{cases} & \begin{cases} +\ 1376, \\ -\ 447, \end{cases} & \begin{cases} +\ 1193, \\ -\ 574, \end{cases} \\
\begin{cases} +\ 1039, \\ -\ 729, \end{cases} & \begin{cases} +\ 884, \\ -\ 884, \end{cases} & &
\end{array}$$

and the stresses in the parts of the booms commencing from the left are

2437·5,	8062·5,	13312·5,	18187·5,
22687·5,	26812·5,	30562·5,	33937·5,
36937·5,	39587·5,	41812·5,	43687·5,
45187·5,	46312·5,	47062·2,	47437·5.

It will appear by comparing these numbers with those given in Fig. 108 that the stresses are much altered when the live load is applied to the lower as well as to the upper joints instead of only to the upper joints.

FOURTH CHAPTER.

§ 15.—Sickle-shaped (Bowstring) Roof of 208 Feet Span with a Single System of Diagonals.

(Roof over Railway Station, Birmingham.)

The unit of length in Fig. 114 is 16 feet, and the dimensions given must therefore be multiplied by 16 to obtain them in feet. Accordingly the span is

$$2 \times 6\cdot 5 \times 16 = 208 \text{ feet,}$$

the height of the upper bow is

$$(1 + 1\cdot 5) \times 16 = 40 \text{ feet,}$$

and that of the lower bow

$$1 \times 16 = 16 \text{ feet,*}$$

and lastly the horizontal length of each bay is

$$1 \times 16 = 16 \text{ feet.}$$

The ordinates of the upper bow are everywhere 2·5 times those of the lower bow.

The load on the roof has been taken at 40 lbs. per foot super of horizontal area covered, including snow and wind-pressure.

The distance apart of the principals being 24 feet, the area supported by each is

$$208 \times 24 = 4992 \text{ sq. feet,}$$

* In the actual roof at Birmingham, the height of the lower bow is 17 feet, but 16 feet has been adopted here to simplify the calculations. Otherwise the dimensions given are the same as those in the actual roof. The calculations have been made for a roof with single diagonals sloping upwards from right to left; but under the head of "Derived Forms" will be found the stresses for a roof similar to the one at Birmingham, with crossed diagonal ties.

and the corresponding load is

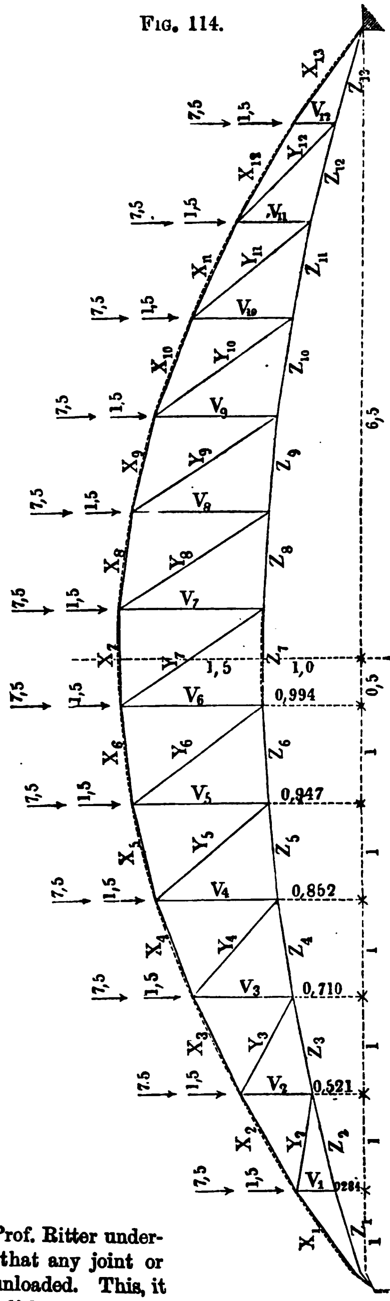
$$4992 \times 40 = 199680 \text{ lbs.}$$

The load on each of the thirteen bays is therefore $\frac{199680}{13} = 15360$ lbs., or $7 \cdot 5$ tons nearly (2000 lbs. to the ton).

The weight of the principal itself deduced from the dimensions of its parts is very nearly $1 \cdot 5$ ton for each bay.

Half the load on each end bay is taken up directly by the abutments, and each of the twelve central joints has $1 \cdot 5$ tons permanent and $7 \cdot 5$ tons variable* load to carry.

FIG. 114.



Calculation of the Stresses X and Z in the Upper and Lower Bows.

Cutting off the part of the roof shown in Fig. 115 by the section line $\alpha\beta$ and taking moments first about M and then about N, the following equations are obtained:

$$0 = X_4 \times 1 \cdot 205 + D \times 4 - 1 \cdot 5(1 + 2 + 3) - 7 \cdot 5(1 + 2 + 3)$$

$$0 = -Z_4 \times 1 \cdot 055 + D \times 3 - 1 \cdot 5(1 + 2) - 7 \cdot 5(1 + 2).$$

*As will be seen in the sequel, Prof. Ritter under-by the load being variable that any joint or y be loaded and the rest unloaded. This, it served, *is not the usual* English practice in the case of roofs.—TRANS.

Substituting for D its value:

$$D = 1 \cdot 5 \left(\tfrac{1}{13} + \tfrac{2}{13} + \ldots + \tfrac{12}{13}\right) + 7 \cdot 5 \left(\tfrac{1}{13} + \tfrac{2}{13} + \ldots + \tfrac{12}{13}\right),$$

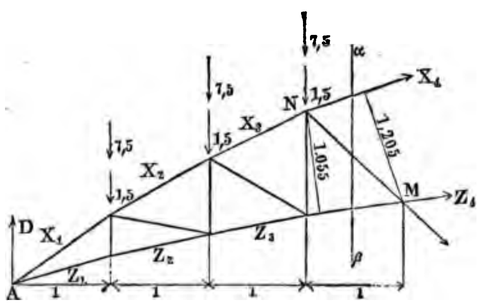
and arranging the equations according to the previous rule so that the effect of the variable load may be traced:

$$\begin{aligned} 0 = {} & X_4 \times 1 \cdot 205 \\ & + 1 \cdot 5 \left\{ \left(\tfrac{1}{13} + \ldots + \tfrac{9}{13}\right) 4 + \left(\tfrac{10}{13} \,.\, 4 - 1\right) + \left(\tfrac{11}{13} \,.\, 4 - 2\right) + \left(\tfrac{12}{13} \,.\, 4 - 3\right) \right\} \\ & + 7 \cdot 5 \left(\tfrac{1}{13} + \ldots + \tfrac{9}{13}\right) 4 \\ & + 7 \cdot 5 \left\{ \left(\tfrac{10}{13} \,.\, 4 - 1\right) + \left(\tfrac{11}{13} \,.\, 4 - 2\right) + \left(\tfrac{12}{13} \,.\, 4 - 3\right) \right. \end{aligned}$$

$$\begin{aligned} 0 = {} & - Z_4 \times 1 \cdot 055 \\ & + 1 \cdot 5 \left\{ \left(\tfrac{1}{13} + \ldots + \tfrac{10}{13}\right) 3 + \left(\tfrac{11}{13} \,.\, 3 - 1\right) + \left(\tfrac{12}{13} \,.\, 3 - 2\right) \right\} \\ & + 7 \cdot 5 \left(\tfrac{1}{13} + \ldots + \tfrac{10}{13}\right) 3 \\ & + 7 \cdot 5 \left\{ \left(\tfrac{11}{13} \,.\, 3 - 1\right) + \left(\tfrac{12}{13} \,.\, 3 - 2\right) \right\} \end{aligned}$$

The members containing the variable load, $7 \cdot 5$ tons, are all positive and therefore of the same sign as the members due to the permanent load $1 \cdot 5$ ton. Hence the stresses X_4 and Z_4 are

FIG. 115.



greatest when the structure is fully loaded. Solving these equations:

$$X_4 \text{ (min.)} = - 134 \cdot 4 \text{ tons.}$$
$$Z_4 \text{ (max.)} = + 128 \cdot 0 \text{ tons.}$$

It having thus been shown that the greatest stresses in the bows occur when the roof is fully loaded, it is better to substitute for D its corresponding value,

$$D = \frac{1 \cdot 5 + 7 \cdot 5}{2} \times 12 = 54 \text{ tons,}$$

in the original equations, which then become

$$0 = X_4 \times 1{\cdot}205 + 54 \times 4 - 9(1 + 2 + 3)$$
$$0 = - Z_4 \times 1{\cdot}055 + 54 \times 3 - 9(1 + 2).$$

In a similar manner the equations for the remaining parts of the bows are obtained as given below:

$$0 = X_1 \times 0{\cdot}347 + 54 \times 1$$
$$X_1 \text{ (min.)} = - 155{\cdot}6 \text{ tons}$$
$$0 = - Z_1 \times 0{\cdot}41 + 54 \times 1$$
$$Z_1 \text{ (max.)} = + 131{\cdot}7 \text{ tons}$$
$$0 = X_2 \times 0{\cdot}672 + 54 \times 2 - 9 \times 1$$
$$X_2 \text{ (min.)} = - 147{\cdot}3 \text{ tons}$$
$$0 = - Z_2 \times 0{\cdot}415 + 54 \times 1$$
$$Z_2 \text{ (max.)} = + 130{\cdot}2 \text{ tons}$$
$$0 = X_3 \times 0{\cdot}963 + 54 \times 3 - 9(1 + 2)$$
$$X_3 \text{ (min.)} = - 140{\cdot}2 \text{ tons}$$
$$0 = - Z_3 \times 0{\cdot}767 + 54 \times 2 - 9 \times 1$$
$$Z_3 \text{ (max.)} = + 129{\cdot}1 \text{ tons}$$
$$0 = X_5 \times 1{\cdot}382 + 54 \times 5 - 9(1 + 2 + 3 + 4)$$
$$X_5 \text{ (min.)} = - 130{\cdot}2 \text{ tons}$$
$$0 = - Z_5 \times 1{\cdot}272 + 54 \times 4 - 9(1 + 2 + 3)$$
$$Z_5 \text{ (max.)} = + 127{\cdot}3 \text{ tons}$$
$$0 = X_6 \times 1{\cdot}481 + 54 \times 6 - 9(1 + 2 + 3 + 4 + 5)$$
$$X_6 \text{ (min.)} = - 127{\cdot}6 \text{ tons}$$
$$0 = - Z_6 \times 1{\cdot}419 + 54 \times 5 - 9(1 + 2 + 3 + 4)$$
$$Z_6 \text{ (max.)} = + 126{\cdot}9 \text{ tons}$$
$$0 = X_7 \times 1{\cdot}491 + 54 \times 7 - 9(1 + 2 + 3 + 4 + 5 + 6)$$
$$X_7 \text{ (min.)} = - 126{\cdot}7 \text{ tons}$$
$$0 = - Z_7 \times 1{\cdot}491 + 54 \times 6 - 9(1 + 2 + 3 + 4 + 5)$$
$$Z_7 \text{ (max.)} = + 126{\cdot}7$$
$$0 = X_8 \times 1{\cdot}41 + 54 \times 8 - 9(1 + 2 + \ldots + 7)$$
$$X_8 \text{ (min.)} = - 127{\cdot}6 \text{ tons}$$
$$0 = - Z_8 \times 1{\cdot}489 + 54 \times 7 - 9(1 + 2 + \ldots + 6)$$
$$Z_8 \text{ (max.)} = + 126{\cdot}9$$
$$0 = X_9 \times 1{\cdot}244 + 54 \times 9 - 9(1 + 2 + \ldots + 8)$$
$$X_9 \text{ (min.)} = - 130{\cdot}2 \text{ tons}$$
$$0 = - Z_9 \times 1{\cdot}414 + 54 \times 8 - 9(1 + 2 + \ldots + 7)$$
$$Z_9 \text{ (max.)} = + 127{\cdot}3 \text{ tons}$$
$$0 = X_{10} \times 1{\cdot}004 + 54 \times 10 - 9(1 + 2 + \ldots + 9)$$
$$X_{10} \text{ (min.)} = - 134{\cdot}4 \text{ tons}$$
$$0 = - Z_{10} \times 1{\cdot}265 + 54 \times 9 - 9(1 + 2 + \ldots + 8)$$
$$Z_{10} \text{ (max.)} = + 128{\cdot}0 \text{ tons}$$

$$0 = X_{11} \times 0{\cdot}706 + 54 \times 11 - 9\,(1 + 2 + \ldots + 10)$$
$$X_{11}\ (\text{min.}) = -\ 140{\cdot}2 \text{ tons}$$

$$0 = -\ Z_{11} \times 1{\cdot}046 + 54 \times 10 - 9\,(1 + 2 + \ldots + 9)$$
$$Z_{11}\ (\text{max.}) = +\ 129{\cdot}1 \text{ tons}$$

$$0 = X_{12} \times 0{\cdot}367 + 54 \times 12 - 9\,(1 + 2 + \ldots + 11)$$
$$X_{12}\ (\text{min.}) = -\ 147{\cdot}3 \text{ tons}$$

$$0 = -\ Z_{12} \times 0{\cdot}76 + 54 \times 11 - 9\,(1 + 2 + \ldots + 10)$$
$$Z_{12}\ (\text{max.}) = +\ 130{\cdot}2 \text{ tons}$$

$$0 = X_{13} \times 0{\cdot}347 + 54 \times 12 - 9\,(1 + 2 + \ldots + 11)$$
$$X_{13}\ (\text{min.}) = -\ 155{\cdot}6 \text{ tons}$$

$$0 = -\ Z_{13} \times 0{\cdot}41 + 54 \times 12 - 9\,(1 + 2 + \ldots + 11)$$
$$Z_{13}\ (\text{max.}) = +\ 131{\cdot}7 \text{ tons.}$$

It appears from these results that the greatest stresses in the symmetrically placed parts of the bow are equal. Now as the only difference between the corresponding bays on each side of the centre is in the direction of the diagonal, it follows that the greatest stresses in the bows are independent of the position of the diagonals. It therefore makes no difference in the results if the point round which moments are taken is at the right or left angle of the bay; that is, whether the point lies in the diagonal or not. But this cannot be the case unless the stress in the diagonal is nothing. From this it follows that the diagonals have no stress in them when the roof is fully loaded.

This property of bowstring roofs will be further discussed in the "Theory of bowstring trusses."

Calculation of the Stress Y in the Diagonals.

To determine the stress Y_4 take moments about O for the part of the roof given in Fig. 116. O is the point of intersection of X_4 and Z_4, and it is found by construction that its distance to the left of A is 2, and that the lever arm of Y_4 is 4·68. Hence the equation,

$$0 = Y_4 \times 4{\cdot}68 - D \times 2 + 1{\cdot}5\,\{(3+2) + (2+2) + (1+2)\}$$
$$+\ 7{\cdot}5\,\{(3+2) + (2+2) + (1+2)\}\,.$$

substituting for D its value

$$D = 1 \cdot 5\left(\tfrac{1}{13} + \tfrac{2}{13} + \ldots + \tfrac{12}{13}\right) + 7 \cdot 5\left(\tfrac{1}{13} + \tfrac{2}{13} + \ldots + \tfrac{12}{13}\right);$$

and arranging the equation so as to be able to observe the effect of the variable load,

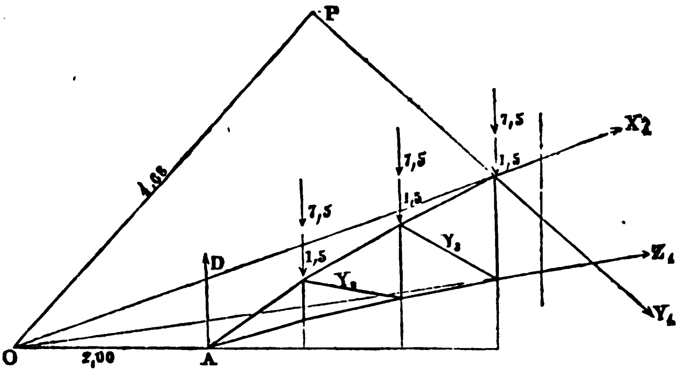
$$\begin{aligned} 0 = Y_4 \times 4 \cdot 68 &+ 1 \cdot 5 \left\{ \left[3 + 2\left(1 - \tfrac{10}{13}\right)\right] + \left[2 + 2\left(1 - \tfrac{11}{13}\right)\right] \right. \\ &\left. + \left[1 + 2\left(1 - \tfrac{12}{13}\right)\right] - \left(\tfrac{1}{13} + \tfrac{2}{13} + \ldots + \tfrac{9}{13}\right) 2 \right\} \\ &+ 7 \cdot 5 \left\{ \left[3 + 2\left(1 - \tfrac{10}{13}\right)\right] + \left[2 + 2\left(1 - \tfrac{11}{13}\right)\right] + \left[1 + 2\left(1 - \tfrac{12}{13}\right)\right] \right\} \\ &- 7 \cdot 5\left(\tfrac{1}{13} + \tfrac{2}{13} + \ldots + \tfrac{9}{13}\right) \times 2, \end{aligned}$$

which simplifies to the following:

$$\begin{aligned} 0 = Y_4 \times 4 \cdot 68 &- 1 \cdot 5 \left[\left(\tfrac{1}{13} + \ldots + \tfrac{9}{13}\right) 2 - (3 + 2 + 1)\left(1 + \tfrac{2}{13}\right)\right] \\ &- 7 \cdot 5\left(\tfrac{1}{13} + \ldots + \tfrac{9}{13}\right) 2 + 7 \cdot 5 (3 + 2 + 1)\left(1 + \tfrac{2}{13}\right). \end{aligned}$$

On calculation it appears that the co-efficient of 1·5 is zero, thus confirming the result obtained above, by means of the greatest stresses in the booms, that a uniformly distributed

FIG. 116.



load, such as the weight of the principal itself, produces no stress in the diagonals. The last equation can therefore be written in the simpler form:

$$0 = Y_4 \times 4 \cdot 68 - 7 \cdot 5\left(\tfrac{1}{13} + \ldots \tfrac{9}{13}\right) 2 + 7 \cdot 5 (3 + 2 + 1)\left(1 + \tfrac{2}{13}\right).$$

Leaving out first the positive then the negative members containing 7·5 tons.

$$Y_4 \begin{cases} (\text{max.}) = +\ 11 \cdot 1 \text{ tons}; \\ (\text{min.}) = -\ 11 \cdot 1 \text{ tons}. \end{cases}$$

This result again shows that a uniformly distributed load produces no stress in the diagonals, for the maximum and minimum values of Y_4 are equal.

The stresses in the remaining diagonals are obtained in a precisely similar manner. The equations given below have been written in the simplest form, that is, omitting the permanent load:

$$0 = Y_2 \times 0{\cdot}92 - 7{\cdot}5\left(\tfrac{1}{13} + \ldots + \tfrac{11}{13}\right) 0{\cdot}2 + 7{\cdot}5\left(1 + \tfrac{0{\cdot}2}{13}\right)$$

$$Y_2 \begin{cases} (\text{max.}) = +\ 8{\cdot}3 \text{ tons} \\ (\text{min.}) = -\ 8{\cdot}3 \text{ tons} \end{cases}$$

$$0 = Y_3 \times 2{\cdot}52 - 7{\cdot}5\left(\tfrac{1}{13} + \ldots + \tfrac{10}{13}\right) 0{\cdot}75 + 7{\cdot}5\,(2 + 1)\left(1 + \tfrac{0{\cdot}75}{13}\right)$$

$$Y_3 \begin{cases} (\text{max.}) = +\ 9{\cdot}5 \text{ tons} \\ (\text{min.}) = -\ 9{\cdot}5 \text{ tons} \end{cases}$$

$$0 = Y_5 \times 8{\cdot}3 - 7{\cdot}5\left(\tfrac{1}{13} + \ldots + \tfrac{8}{13}\right) 5 + 7{\cdot}5\,(4 + 3 + 2 + 1)\left(1 + \tfrac{5}{13}\right)$$

$$Y_5 \begin{cases} (\text{max.}) = +\ 12{\cdot}6 \text{ tons} \\ (\text{min.}) = -\ 12{\cdot}6 \text{ tons} \end{cases}$$

$$0 = Y_6 \times 17{\cdot}6 - 7{\cdot}5\left(\tfrac{1}{13} + \ldots + \tfrac{7}{13}\right) 15 + 7{\cdot}5\,(5 + 4 + 3 + 2 + 1)\left(1 + \tfrac{15}{13}\right)$$

$$Y_6 \begin{cases} (\text{max.}) = +\ 13{\cdot}8 \text{ tons} \\ (\text{min.}) = -\ 13{\cdot}8 \text{ tons.} \end{cases}$$

The point about which to take moments for the diagonal in the central bay lies at infinity; it is therefore necessary to follow the rule given in the third section, § 9. The sine of the angle this diagonal makes with the horizontal is $= 0{\cdot}831$.

The equation to find Y_7 is therefore,

$$0 = Y_7 \times 0{\cdot}831 \times \infty - 7{\cdot}5\left(\tfrac{1}{13} + \ldots + \tfrac{6}{13}\right) \infty + 7{\cdot}5\,(6 + 5 + 4 + 3 + 2 + 1)\left(1 + \tfrac{\infty}{13}\right);$$

or since the finite vanishes in comparison to the infinite,

$$0 = Y_7 \times 0{\cdot}831 \times \infty - 7{\cdot}5\left(\tfrac{1}{13} + \ldots + \tfrac{6}{13}\right) \infty + 7{\cdot}5\,(6 + 5 + 4 + 3 + 2 + 1)\,\tfrac{\infty}{13};$$

and dividing out by the common factor ∞,

$$0 = Y_7 \times 0{\cdot}831 - 7{\cdot}5\left(\tfrac{1}{13} + \ldots + \tfrac{6}{13}\right) + 7{\cdot}5\,(6 + \ldots + 1)\,\tfrac{1}{13}$$

$$Y_7 \begin{cases} (\text{max.}) = +\ 14{\cdot}6 \text{ tons} \\ (\text{min.}) = -\ 14{\cdot}6 \text{ tons} \end{cases}$$

In the following equations the points about which moments are taken fall to the right of the section line:

$$0 = -\,Y_8 \times 16{\cdot}1 + 7{\cdot}5\left(\tfrac{1}{13} + \ldots + \tfrac{5}{13}\right) 28 - 7{\cdot}5\,(7 + \ldots + 1)\left(\tfrac{28}{13} - 1\right)$$

$$Y_8 \begin{cases} (\text{max.}) = +\ 15{\cdot}0 \text{ tons} \\ (\text{min.}) = -\ 15{\cdot}0 \text{ tons} \end{cases}$$

$$0 = -Y_9 \times 7 \cdot 1 + 7 \cdot 5\left(\tfrac{1}{13} + \ldots + \tfrac{4}{13}\right) 18$$
$$- 7 \cdot 5\,(8 + \ldots + 1)\left(\tfrac{18}{13} - 1\right)$$
$$Y_9 \begin{cases} \text{(max.)} = +\,14 \cdot 6 \text{ tons} \\ \text{(min.)} = -\,14 \cdot 6 \text{ tons} \end{cases}$$

$$0 = -Y_{10} \times 3 \cdot 68 + 7 \cdot 5\left(\tfrac{1}{13} + \tfrac{2}{13} + \tfrac{3}{13}\right) 15$$
$$- 7 \cdot 5\,(9 + \ldots + 1)\left(\tfrac{15}{13} - 1\right)$$
$$Y_{10} \begin{cases} \text{(max.)} = +\,14 \cdot 1 \text{ tons} \\ \text{(min.)} = -\,14 \cdot 1 \text{ tons} \end{cases}$$

$$0 = -Y_{11} \times 1 \cdot 82 + 7 \cdot 5\left(\tfrac{1}{13} + \tfrac{2}{13}\right) 13 \cdot 75$$
$$- 7 \cdot 5\,(10 + \ldots + 1)\left(\frac{13 \cdot 75}{13} - 1\right)$$
$$Y_{11} \begin{cases} \text{(max.)} = +\,13 \cdot 0 \text{ tons} \\ \text{(min.)} = -\,13 \cdot 0 \text{ tons} \end{cases}$$

$$0 = -Y_{12} \times 0 \cdot 65 + 7 \cdot 5 \times \tfrac{1}{13} \times 13 \cdot 2$$
$$- 7 \cdot 5\,(11 + \ldots + 1)\left(\frac{13 \cdot 2}{13} - 1\right)$$
$$Y_{12} \begin{cases} \text{(max.)} = +\,11 \cdot 6 \text{ tons} \\ \text{(min.)} = -\,11 \cdot 6 \text{ tons.} \end{cases}$$

Calculation of the Stresses V in the Verticals.

To find V_1 take moments for the part of the roof shown in Fig. 117, round the point of intersection of X_1 and Z_2. By

FIG. 117.

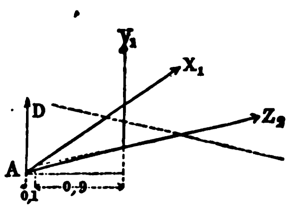
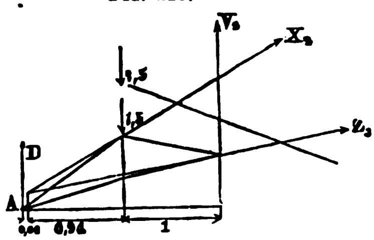


FIG. 118.



construction it is found that this point is at a horizontal distance of $0 \cdot 1$ to the right of A. Hence the equation of moments is:

$$0 = -V_1 \times 0 \cdot 9 + D \times 0 \cdot 1,$$

or substituting for D and arranging the equation,

$$0 = -V_1 \times 0 \cdot 9 + 1 \cdot 5\left(\tfrac{1}{13} + \tfrac{2}{13} + \ldots + \tfrac{12}{13}\right) 0 \cdot 1$$
$$+ 7 \cdot 5\left(\tfrac{1}{13} + \tfrac{2}{13} + \ldots + \tfrac{12}{13}\right) 0 \cdot 1.$$

The co-efficient of $7 \cdot 5$ is positive, and therefore V_1 is greatest when the variable load covers the roof. Solving the equation

$$V_1 \text{ (max.)} = +\,6 \text{ tons.}$$

The stress V_2 is to be found from Fig. 118. The point of

intersection of X_2 and Z_3 is at a horizontal distance of 0·06 to the right of A and the equation of moments is

$$0 = -V_2 \times 1{\cdot}94 + D \times 0{\cdot}06 + 1{\cdot}5 \times 0{\cdot}94 + 7{\cdot}5 \times 0{\cdot}94,$$

substituting for D and arranging the equation,

$$0 = -V_2 \times 1{\cdot}94 + 1{\cdot}5\left[\left(\tfrac{1}{18} + \ldots + \tfrac{14}{18}\right)0{\cdot}06 + \left(1 - \tfrac{0{\cdot}06}{18}\right)\right]$$
$$+ 7{\cdot}5\left(\tfrac{1}{18} + \ldots + \tfrac{14}{18}\right)0{\cdot}06 + 7{\cdot}5\left(1 - \tfrac{0{\cdot}06}{18}\right).$$

Here again the co-efficient of 7·5 is positive, and the equation can therefore be solved as it stands, whence,

$$V_2\,(\text{max.}) = +\,6 \text{ tons.}$$

For the next vertical, V_3, the point about which to take moments falls to the left of A, and the equation consequently alters its form to a slight extent.

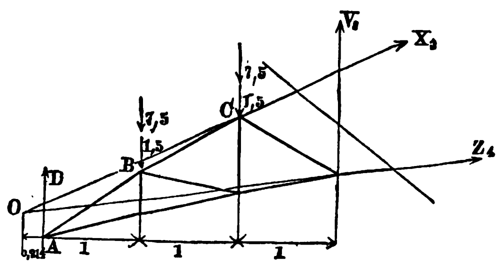
The equation of moments round O in Fig. 119 is

$$0 = -V_3 \times 3{\cdot}214 - D \times 0{\cdot}214 + 1{\cdot}5(1{\cdot}214 + 2{\cdot}214)$$
$$+ 7{\cdot}5\,(1{\cdot}214 + 2{\cdot}214),$$

and substituting for D and arranging the equation

$$0 = -V_3 \times 3{\cdot}214 - 1{\cdot}5\left[\left(\tfrac{1}{18} + \ldots + \tfrac{13}{18}\right)0{\cdot}214 - (2 + 1)\left(1 + \tfrac{0{\cdot}214}{18}\right)\right]$$
$$- 7{\cdot}5\left(\tfrac{1}{18} + \ldots + \tfrac{13}{18}\right)0{\cdot}214 + 7{\cdot}5(2 + 1)\left(1 + \tfrac{0{\cdot}214}{18}\right).$$

FIG. 119.



In this case one of the members containing 7·5 is positive and the other negative. To determine V_3 (max.) the negative member, and to find V_3 (min.), the positive member must be left out. The value of V_3, keeping both these members in the equation, will also be calculated; this gives the value of V_3 when the roof is fully loaded: the reason for doing this will r further on.

The values obtained are

$$V_3\begin{cases}(\text{max.}) = +\ 8\cdot1 \text{ tons.}\\(\text{min.}) = -\ 1\cdot1 \text{ tons.}\end{cases}\qquad V_3 = +\ 6 \text{ tons.}$$

Similarly the following equations are obtained:

$$0 = -\ V_4 \times 4\cdot91 - 1\cdot5\left[\left(\tfrac{1}{13} + \ldots + \tfrac{3}{13}\right)0\cdot91 - (3+2+1)\left(1 + \tfrac{0\cdot91}{13}\right)\right]$$
$$-\ 7\cdot5\left(\tfrac{1}{13} + \ldots + \tfrac{3}{13}\right)0\cdot91 + 7\cdot5\,(3+2+1)\left(1 + \tfrac{0\cdot91}{13}\right)\Big].$$

$$V_4\begin{cases}(\text{max.}) = +\ 10\cdot8 \text{ tons.}\\(\text{min.}) = -\ 3\cdot8 \text{ tons.}\end{cases}\qquad V_4 = 6 \text{ tons.}$$

$$0 = -\ V_5 \times 7\cdot5 - 1\cdot5\left[\left(\tfrac{1}{13} + \ldots + \tfrac{4}{13}\right)2\cdot5 - (4 + \ldots + 1)\left(1 + \tfrac{2\cdot5}{13}\right)\right]$$
$$-\ 7\cdot5\left(\tfrac{1}{13} + \ldots + \tfrac{4}{13}\right)2\cdot5 + 7\cdot5\,(4 + \ldots + 1)\left(1 + \tfrac{2\cdot5}{13}\right)\Big].$$

$$V_5\begin{cases}(\text{max.}) = +\ 12\cdot9 \text{ tons.}\\(\text{min.}) = -\ 5\cdot9 \text{ tons.}\end{cases}\qquad V_5 = +\ 6 \text{ tons.}$$

$$0 = -\ V_6 \times 12\cdot6 - 1\cdot5\left[\left(\tfrac{1}{13} + \ldots + \tfrac{5}{13}\right)6\cdot6 - (5 + \ldots + 1)\left(1 + \tfrac{6\cdot6}{13}\right)\right]$$
$$-\ 7\cdot5\left(\tfrac{1}{13} + \ldots + \tfrac{5}{13}\right)6\cdot6 + 7\cdot5\,(5 + \ldots + 1)\left(1 + \tfrac{6\cdot6}{13}\right)\Big].$$

$$V_6\begin{cases}(\text{max.}) = +\ 14\cdot5 \text{ tons.}\\(\text{min.}) = -\ 7\cdot5 \text{ tons.}\end{cases}\qquad V_6 = +\ 6 \text{ tons.}$$

$$0 = -\ V_7 \times 31\cdot5 - 1\cdot5\left[\left(\tfrac{1}{13} + \ldots + \tfrac{6}{13}\right)24\cdot5 - (6 + \ldots + 1)\left(1 + \tfrac{24\cdot5}{13}\right)\right]$$
$$-\ 7\cdot5\left(\tfrac{1}{13} + \ldots + \tfrac{6}{13}\right)24\cdot5 + 7\cdot5\,(6 + \ldots + 1)\left(1 + \tfrac{24\cdot5}{13}\right)\Big].$$

$$V_7\begin{cases}(\text{max.}) = +\ 15\cdot4 \text{ tons.}\\(\text{min.}) = -\ 8\cdot4 \text{ tons.}\end{cases}\qquad V_7 = +\ 6 \text{ tons.}$$

In the remaining bays the point about which moments are taken is to the right of the section line, and the signs of the equations are consequently changed.

$$0 = V_8 \times 60 + 1\cdot5\left[\left(\tfrac{1}{13} + \ldots + \tfrac{7}{13}\right)68 - (7 + \ldots + 1)\left(\tfrac{68}{13} - 1\right)\right]$$
$$+\ 7\cdot5\left(\tfrac{1}{13} + \ldots + \tfrac{7}{13}\right)68 - 7\cdot5\,(7 + \ldots + 1)\left(\tfrac{68}{13} - 1\right).$$

$$V_8\begin{cases}(\text{max.}) = +\ 15\cdot8 \text{ tons.}\\(\text{min.}) = -\ 8\cdot8 \text{ tons.}\end{cases}\qquad V_8 = +\ 6 \text{ tons.}$$

$$0 = V_9 \times 13\cdot5 + 1\cdot5\left[\left(\tfrac{1}{13} + \ldots + \tfrac{4}{13}\right)22\cdot5 - (8 + \ldots + 1)\left(\tfrac{22\cdot5}{13} - 1\right)\right]$$
$$+\ 7\cdot5\left(\tfrac{1}{13} + \ldots + \tfrac{4}{13}\right)22\cdot5 - 7\cdot5\,(8 + \ldots + 1)\left(\tfrac{22\cdot5}{13} - 1\right).$$

$$V_9\begin{cases}(\text{max.}) = +\ 15\cdot6 \text{ tons.}\\(\text{min.}) = -\ 8\cdot6 \text{ tons.}\end{cases}\qquad V_9 = +\ 6 \text{ tons.}$$

$$0 = V_{10} \times 6\cdot43 + 1\cdot5\left[\left(\tfrac{1}{13} + \tfrac{2}{13} + \tfrac{3}{13}\right)16\cdot43 - (9 + \ldots + 1)\left(\tfrac{16\cdot43}{13} - 1\right)\right]$$
$$+\ 7\cdot5\left(\tfrac{1}{13} + \tfrac{2}{13} + \tfrac{3}{13}\right)16\cdot43 - 7\cdot5\,(9 + \ldots + 1)\left(\tfrac{16\cdot43}{13} - 1\right)\Big].$$

$$V_{10}\begin{cases}(\text{max.}) = +\ 14\cdot8 \text{ tons.}\\(\text{min.}) = -\ 7\cdot8 \text{ tons.}\end{cases}\qquad V_{10} = +\ 6 \text{ tons.}$$

$$0 = V_{11} \times 3\cdot3 + 1\cdot5\left[\left(\tfrac{1}{13} + \tfrac{2}{13}\right)14\cdot3 - (10 + \ldots + 1)\left(\tfrac{14\cdot3}{13} - 1\right)\right]$$
$$+\ 7\cdot5\left(\tfrac{1}{13} + \tfrac{2}{13}\right)14\cdot3 - 7\cdot5\,(10 + \ldots + 1)\left(\tfrac{14\cdot3}{13} - 1\right).$$

$$V_{11}\begin{cases}(\text{max.}) = +\ 13\cdot5 \text{ tons.}\\(\text{min.}) = -\ 6\cdot5 \text{ tons.}\end{cases}\qquad V_{11} = +\ 6 \text{ tons.}$$

$$0 = V_{12} \times 1\cdot385 + 1\cdot5\left[\tfrac{1}{13} \times 13\cdot385 - (11 + \ldots + 1)\left(\tfrac{13\cdot385}{13} - 1\right)\right]$$
$$+\ 7\cdot5 \times \tfrac{1}{13} \times 13\cdot385 - 7\cdot5(11 + \ldots + 1)\left(\tfrac{13\cdot385}{13} - 1\right).$$

It was recommended in § 12 to assume that both the permanent and variable loads were applied to the same joints, and this assumption was made possible by the introduction of secondary verticals, whose object was to convey to the supposed loaded joints the part of the permanent load belonging to the other joints. In the present case it was supposed that the whole of the weight of the principal was applied to the top joints. Now in reality, this load is distributed between the upper and lower joints, but the upper joints have the greater proportion to bear, and only about one-third or $0\cdot5$ ton of the permanent load on each bay falls directly on each lower joint. The secondary vertical introduced to transmit this load to the upper joints is therefore a tie, and the tension in it is $0\cdot5$ ton, and this stress must be added to the stresses in the verticals previously found.

The more accurate values of the stresses in the verticals are therefore

$$V_1\,(\text{max.}) = +\ 6\cdot5 \text{ tons.}$$
$$V_2\,(\text{max.}) = +\ 6\cdot5 \text{ tons.}$$

$$V_3\begin{cases}(\text{max.}) = +\ 8\cdot6 \text{ tons.}\\(\text{min.}) = -\ 0\cdot6 \text{ tons.}\end{cases}\qquad V_3 = +\ 6\cdot5 \text{ tons.}$$

$$V_4\begin{cases}(\text{max.}) = +\ 11\cdot3 \text{ tons.}\\(\text{min.}) = -\ 3\cdot3 \text{ tons.}\end{cases}\qquad V_4 = +\ 6\cdot5 \text{ tons.}$$

$$V_5\begin{cases}(\text{max.}) = +\ 13\cdot4 \text{ tons.}\\(\text{min.}) = -\ 5\cdot4 \text{ tons.}\end{cases}\qquad V_5 = +\ 6\cdot5 \text{ tons.}$$

$$V_6\begin{cases}(\text{max.}) = +\ 15\cdot0 \text{ tons.}\\(\text{min.}) = -\ 7\cdot0 \text{ tons.}\end{cases}\qquad V_6 = +\ 6\cdot5 \text{ tons.}$$

$$V_7\begin{cases}(\text{max.}) = +\ 15\cdot9 \text{ tons.}\\(\text{min.}) = -\ 7\cdot9 \text{ tons.}\end{cases}\qquad V_7 = +\ 6\cdot5 \text{ tons.}$$

$$V_8\begin{cases}(\text{max.}) = +\ 16\cdot3 \text{ tons.}\\(\text{min.}) = -\ 8\cdot3 \text{ tons.}\end{cases}\qquad V_8 = +\ 6\cdot5 \text{ tons.}$$

$$V_9\begin{cases}(\text{max.}) = +\ 16\cdot1 \text{ tons.}\\(\text{min.}) = -\ 8\cdot1 \text{ tons.}\end{cases}\qquad V_9 = +\ 6\cdot5 \text{ tons.}$$

$$V_{10}\begin{cases}(\text{max.}) = +\ 15\cdot3 \text{ tons.}\\(\text{min.}) = -\ 7\cdot3 \text{ tons.}\end{cases}\qquad V_{10} = +\ 6\cdot5 \text{ tons.}$$

$$V_{11}\begin{cases}(\text{max.}) = +\,14\cdot 0 \text{ tons.}\\ (\text{min.}) = -\,6\cdot 0 \text{ tons.}\end{cases} \quad V_{11} = +\,6\cdot 5 \text{ tons.}$$

$$V_{12}\begin{cases}(\text{max.}) = +\,12\cdot 1 \text{ tons.}\\ (\text{min.}) = -\,4\cdot 1 \text{ tons.}\end{cases} \quad V_{12} = = 6\cdot 5 \text{ tons.}$$

The whole of the results are given in Fig. 120.

§ 16.—Derived Forms.

The above calculations show that the diagonals of a double bowstring roof possessing only one single system of diagonals are subject both to tension and compression. On examining the equation of moments for the stress in a diagonal it will be seen that the maximum stress in it is reached when all the joints to the right, and the minimum when all the joints to the left of it, are loaded.

If the diagonals were inclined upwards from left to right, the reverse would obviously be the case, and the stresses that then obtain can easily be found by looking at Fig. 120 as it were from behind; or, what amounts to the same thing, the stress in a diagonal inclined upwards from left to right can be found from that in the diagonal situated in the symmetrically placed bay and inclined upwards from right to left.

If in any bay the diagonal inclined to the left can only take up tension, a second diagonal of like properties inclined to the right must be introduced, and it will come into play only when the first one is slack, and *vice versâ*. The stresses in these diagonals can be obtained from Fig. 120; the maximum

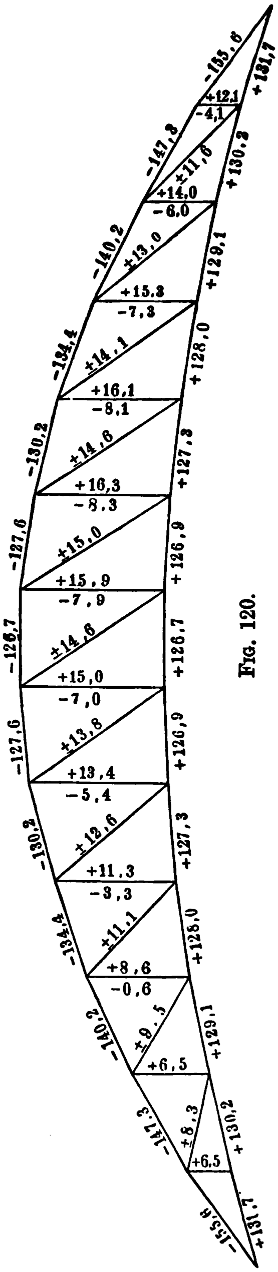
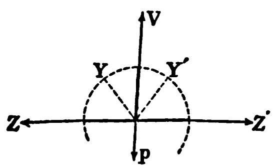


Fig. 120.

stress in the diagonal inclined to the left can be found directly, and the maximum stress in the diagonal inclined to the right will be the same as that in the diagonal of the symmetrically placed bay.

Before the stresses in the verticals of a roof with crossed diagonals can be determined, it is necessary to ascertain which of the diagonals is in tension under the partial loading, for the section line must be parallel to the diagonal which is in tension in order to cut through only three bars. When the roof is fully loaded, the stress in all the diagonals is zero, and at the same time the stress in the lower boom is greatest. The tension in the verticals will then also be greatest. For besides the permanent load p (Fig. 121) the stresses Z and Z′ are the only forces that can produce tension in the vertical. The stresses Y and Y′ in the diagonals, when they exist, produce on the contrary compression, for the resolved parts vertically of the stresses in them act upwards. But with a full load Z and Z′ are greatest, and Y and Y′ are nothing, and therefore the tension in the verticals is greatest under these circumstances.

FIG. 121.



The above appears even more clearly by observing the effect produced by *unloading* one of the joints when the full load is applied. Unloading a joint can be considered as the application of a vertical force acting upwards; and since the diagonals are under no stress when the structure is not loaded at all, as well as when it is fully loaded, it follows that it is only necessary to investigate the effect of a vertical force acting upwards on the unloaded and weightless structure as represented in Fig. 122. The vertical force K produces the reactions D and W at the abutments A and B, and for simplicity only those diagonals have been shown which are brought into tension by this force. To find which of the diagonals in any bay is in tension, take a section through this bay, and form the equation of moments for the part (Fig. 123) which does not contain K, ιd O, the point of intersection of the directions of the

booms. It is then easy to see which of the diagonals must be acting to maintain equilibrium. For instance, in the fourth bay it is the diagonal inclined to the right, for the equation of moments is (Fig. 123)

$$0 = \mathrm{D}d - \mathrm{Y}y \text{ or } \mathrm{Y} = +\frac{\mathrm{D}d}{y}.$$

The stress Y is therefore positive, or the diagonal is in tension

FIG. 122.

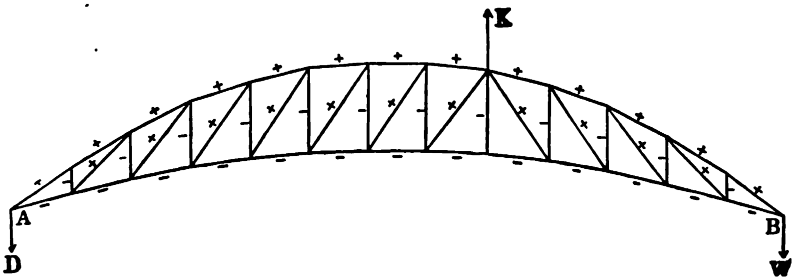
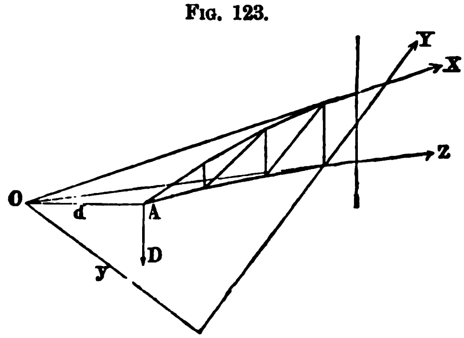


FIG. 123.



(the equation for the other diagonal would give a negative stress or compression).

As soon as it has been determined by this means which of the diagonals are in tension it can be decided by a similar process whether any particular vertical is in tension or compression. Thus, for instance, for the third vertical (Fig. 124) the equation of moments is

$$0 = \mathrm{D}\delta + \mathrm{V}v.$$

from which a negative value is obtained for V, showing that it is in compression.

A different process must, however, be employed for the

vertical, acted on directly by the force K, because the section line would cut through four bars. Now it is easy to see that K produces compression in every part of the lower bow, the equation of moments to find Z (Fig. 125) being

$$0 = -\mathrm{D}l - \mathrm{Z}z \text{ or } \mathrm{D} = -\frac{\mathrm{Z}z}{l}.$$

Thus the parts of the bow acting on the foot of the vertical in question (Fig. 122) being in compression will produce compression in it.

FIG. 124.

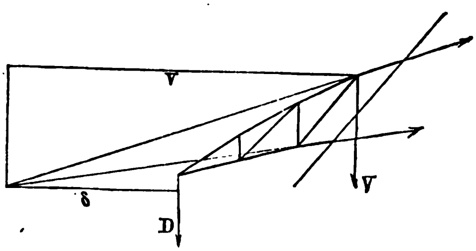
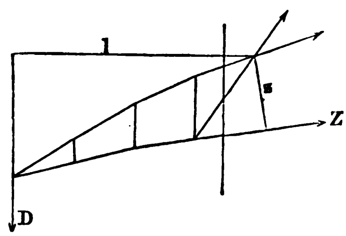


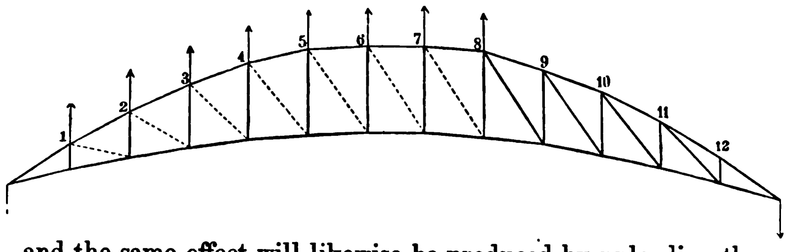
FIG. 125.



It is thus seen that unloading any joint diminishes the stress in all the verticals, from which it follows that the tension in the verticals will be greatest when all the joints are loaded.

It now remains to be decided what joints should be unloaded in order that the stress in any vertical may be a minimum. Take, for instance, the ninth vertical; it is evident that unloading the eighth joint will diminish the stress in it,

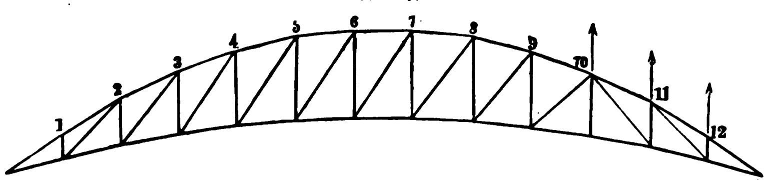
FIG. 126.



and the same effect will likewise be produced by unloading the seventh, sixth, &c., to first joint; and, what is an important point, the unloading of each of these joints will bring the same

system of diagonals into tension in the two bays adjacent to the ninth vertical (Fig. 126). But if the unloading were still further continued the compression in the ninth vertical would be diminished. For since in the two adjacent bays to this vertical the same set of diagonals is in tension (the system inclined to the left), the equation already found for V_9 (min.) holds good, and this equation shows that the compression is diminished by unloading the ninth, tenth, eleventh, and twelfth joints. Hence the value of V_9 (min.) found above is also true if the diagonals are crossed. There is, however, a second minimum value of V_9; for it can be shown in a similar manner to the above that the stress in V_9 is a minimum when the joints 10, 11, and 12 are unloaded and the remainder loaded (Fig. 127). Evidently in this case the ninth vertical is in the same condition as the fourth vertical in Fig. 120, and therefore the value of V_4 (min.) obtains. Hence to find the greatest

FIG. 127.



compression in any vertical the values of the two minima must be compared and the one whose absolute value is greatest taken.

As regards the stresses in the bows, they are greatest when the roof is fully loaded, and consequently when the stress in the diagonals is nothing; the arrangement of the diagonals can therefore produce no alteration in the stresses in the bows.

Thus without any new calculations the stresses already found can be inscribed in Fig. 128, showing a bowstring roof with crossed diagonal ties.

By similar reasoning it is easy to prove that in the case of crossed diagonal struts which are not capable of taking up tension (this is the case in wooden structures) only the maxima values of the stress in the verticals apply, and that in fact

compression cannot occur in the verticals owing to the compression in the diagonals. The stresses given in Fig. 129 require, therefore, no further comment.

It must however be observed, and this does not only apply to this case but also to wherever crossed diagonals exist, that the stresses found above are only true if no artificial stresses exist in the bars. Such artificial stresses cannot occur in single diagonal systems, for in this case every bar can be reached by a

Fig. 128.

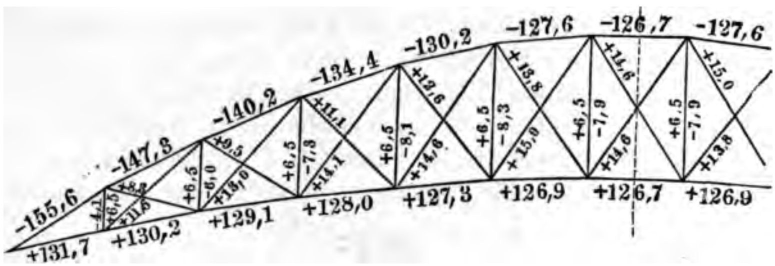
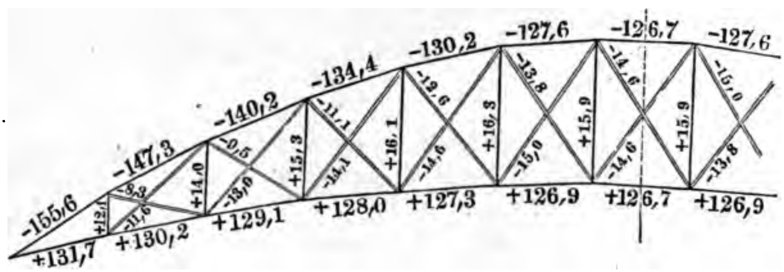


Fig. 129.



section line cutting through only three bars. Thus if no exterior forces are acting on the system, the equation of moments for any bar, whose stress is Y, about the point of intersection of the other two bars included in the section is

$$0 = Y y.$$

But if two diagonals cross each other in a quadrilateral the section line must cut through four bars, and the stresses in the two diagonals tend to turn the part cut off in opposite directions round the point of intersection of the other two bars.

From Fig. 130 the equation of moments is

$$0 = Y y - Y' y',$$

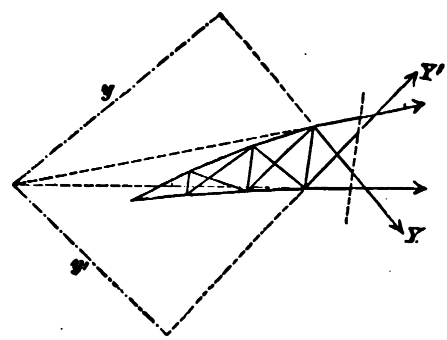
which implies the condition

$$\frac{Y}{Y'} = \frac{y'}{y},$$

but the absolute values of Y and Y′ are indeterminate.

Therefore, if by means of set screws or otherwise an artificial stress Y is set up in one diagonal, the stress in the other will immediately change, in the above proportion, to Y′. This will alter the stresses in the verticals and parts of

FIG. 130.



the bow in the same bay, and they can easily be found by the method of moments as soon as Y is known and Y′ determined.

The stresses given above are therefore only true if, when the structure is unloaded, all the bars are without stress. Then only one of the diagonals (either a tie or a strut) will be acting at any time, but if artificial stresses are introduced it might happen that both diagonals would be acting at the same time.

§ 17.—APPARENT FAILURES OF THE METHOD OF MOMENTS.

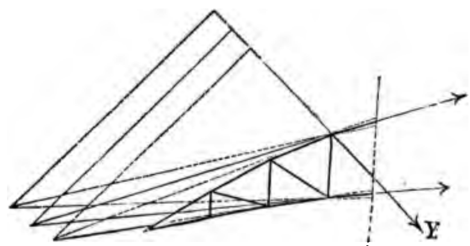
There are cases in the employment of this method, and some have occurred in the last example, in which it would appear that although a result is obtained it can only be approximate.

In every case the point about which moments are taken is the intersection of two of the bars cut through by the section line. When these bars are nearly parallel the accurate determination of this point and the measurement of the lever arms is connected with difficulties. In all probability two distinct computors would arrive at different results.

This would seem to be a great disadvantage of the method. But on further consideration it will appear that it is possible on the contrary to derive some use from the circumstance.

It is clear that limits to the error can be obtained by first intentionally giving the lines too great and secondly too small a convergence (Fig. 131), and calculating in each case the stress. Thus two values are obtained, and evidently the true value

FIG. 131.



lies between them. By comparing these values with the intentionally committed errors it is possible to ascertain to what degree the stresses will be altered by small errors in the carrying out of the work.

For the uncertainty apparent on the drawing is in reality a representation of what actually occurs by errors in the construction. As the workman deviates in one direction or the other from the working drawings, so the stresses will approach one or the other limit. Therefore it is possible to ascertain the alterations produced in the stresses owing to inaccuracies in the carrying out of the design.

A second objection, even less founded than the former, is that the method does not depend entirely on calculation, but must obtain some of its data by graphic means. But calcula-

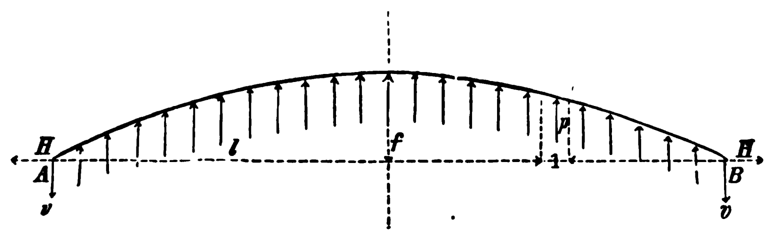
tions should only be made when they arrive quicker at the result than other methods. If, therefore, the graphic method is shorter than calculation it should be adopted, especially as there is less liability to error in measuring than in calculating.

§ 18.—Theory of Sickle-shaped Trusses.

It will be noticed that in the preceding numerical example the stresses were obtained without knowing anything of the laws respecting the distribution of the stresses in the structure. If, however, it were required to determine the form of the structure, it would be necessary to be acquainted with these laws. For this reason it is proposed to extend the "Theory of Parabolic Trusses," commenced in § 8.

In that paragraph the equilibrium of a loaded chain was considered (Fig. 42). If this chain be imagined to rotate through two right angles about the horizontal axis A B, the vertical

Fig. 132.



forces will be reversed in direction and Fig. 132 obtained. The chain can be considered as negatively loaded, and evidently the equation obtained for Fig. 42 remains true, namely:

$$H f = \frac{p\,l^2}{2}.$$

Similarly for another parabolic chain (Fig. 133) loaded with a positive load P per unit of length of the span, the equation

$$H F = \frac{P\,l^2}{2}$$

holds good, and evidently the load P can be so chosen that the horizontal thrust H will be the same as the horizontal pull H

in the former case. The requisite condition can be obtained by dividing one equation by the other, thus:

$$\frac{f}{F} = \frac{p}{P},$$

or in words: the loads per unit of length of the span must be as the heights of arc.

If both these chains are placed on the same abutments (Fig. 134) the reaction will be entirely vertical, for the hori-

FIG. 133.

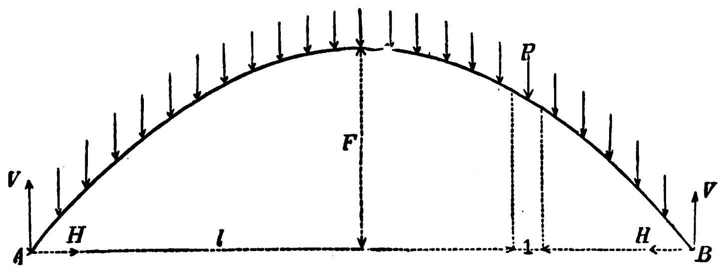
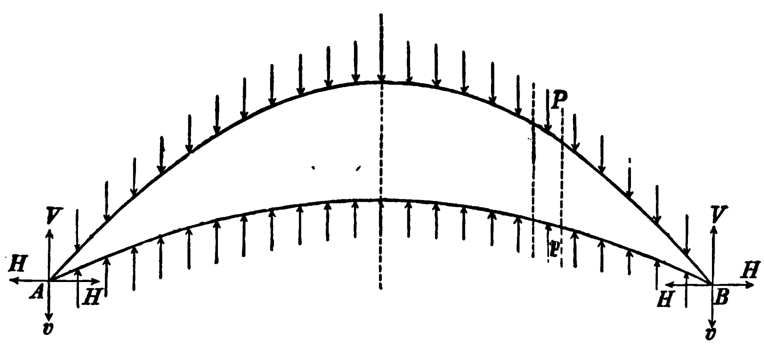


FIG. 134.



zontal thrust of one chain neutralizes the horizontal pull of the other. The vertical reaction at the abutment will therefore be equal to

$$V - v = P\,l - p\,l = (P - p)\,l.$$

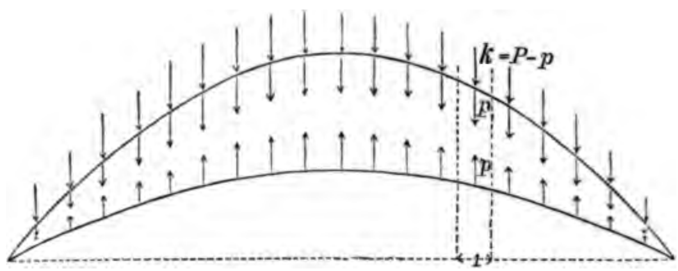
This reaction is equal to that produced by a straight girder of span $2\,l$ and with a uniformly distributed load of $P - p$.

The lower chain can be loaded negatively by means of ties

pulling upwards, the tension in them being equal to p per unit of length of span.

In the same manner part of the positive load on the upper chain can be produced by means of ties pulling downwards. If this part of the load be equal to the negative load on the lower chain, namely p per unit of length of the span, there will still be a load $P - p$ on the upper chain, which will be designated by k and which can be applied by external loads (Fig. 135); if the

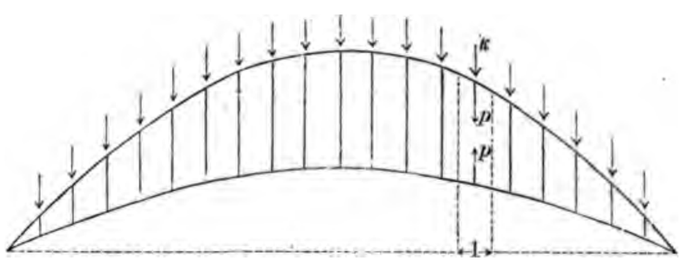
FIG. 135.



ties of the upper and lower chains be considered joined together, the load p can be omitted, for its effect is exactly reproduced by these vertical ties. A double bowstring truss without diagonals has thus been built up, carrying a load on the top equal to k per unit of length of the span.

The stresses in both bows as well as in the verticals can be

FIG. 136.



calculated from the magnitudes l, f, F, k. For simplicity there is a vertical to every unit of length of the span (Fig. 136), and this is quite legitimate, for it was shown in § 8 that the load could be concentrated at points, and so long as the load at

each point was equal to half the uniform load on the adjoining intervals those points remained on the parabola.

P and p can be found from the equations;

$$\frac{f}{F} = \frac{p}{P}, \text{and } P - p = K.$$

Putting $\frac{f}{F} = \frac{1}{n}$ and reducing,

$$p = \frac{k}{n-1}, P = k + \frac{k}{n-1}.$$

The load k produces therefore a tension $\frac{k}{n-1}$ in the verticals.

In the preceding numerical example

$$\frac{1}{n} = \frac{f}{F} = \frac{1}{2 \cdot 5} = \frac{2}{5} \therefore n = \frac{5}{2}.$$

Hence the tension in the verticals (or the negative load on the lower bow) is in this case

$$p = \frac{k}{\frac{5}{2} - 1} = \frac{2}{3} k,$$

and the load on the upper bow is

$$P = k + p = \frac{5}{3} . k.$$

Thus, if the external load on each top joint is $7 \cdot 5$ tons, the tension in each vertical will be $\frac{2}{3} \times 7 \cdot 5 = 5$ tons, and the upper bow is in the same condition as if loaded with $\frac{5}{3} \times 7 \cdot 5 = 12 \cdot 5$ tons at each joint.

The load of $1 \cdot 5$ tons on each top joint due to the weight of the truss itself produces a tension in the verticals $= \frac{2}{3} \times 1.5 = 1$ ton, and the positive load on the upper bow is $\frac{5}{3} \times 1 \cdot 5 = 2 \cdot 5$ tons.

Lastly, if $7 \cdot 5 + 1 \cdot 5 = 9$ tons is the total load on each top joint, the negative load per unit of length of span on the lower bow is 6 tons, the positive load on the upper bow is 15 tons, and the tension in each vertical is 6 tons.

If, however, part of the load is applied at the lower joints it must be conveyed by secondary verticals to the top joints, and the tension in these secondary verticals is to be added to that in the main verticals. For instance, in the preceding example

0·5 ton of the weight of the truss was considered as acting on the lower joints; 0·5 ton must therefore be added to the 6 tons tension found above, and this coincides exactly with the value obtained by the method of moments. The negative load on the lower bow remains the same as before, namely 6 tons, for the tension in the secondary verticals evidently does not affect it.

The constant horizontal stress in the bows is:

$$H + \frac{p\,l^2}{2f} = \frac{6 \times (6\cdot5)^2}{2 \times 1} = 126\cdot7 \text{ tons},$$

which is tension in the lower bow and compression in the upper bow. This is the same value that was obtained by the method of moments ($Z_7 = +\ 126\cdot7$ tons, and $X_7 = -\ 126\cdot7$ tons).

If $\frac{1}{n} = 0$, it follows that $p = 0$ and $P = k$; that is, if the lower bow becomes a horizontal straight line the loading of the upper bow produces no tension in the verticals.

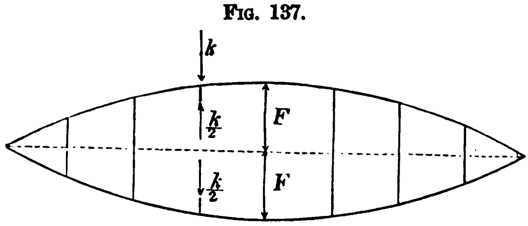
Further, if $\frac{1}{n}$ becomes negative p also becomes negative; that is, the loading of the upper bow produces compression in the verticals. For instance, if

$$\frac{1}{n} = -1,$$

$$p = -\frac{k}{2} \text{ and } P = +\frac{k}{2}.$$

In this case, therefore, one half of the load placed on the top is transferred to the bottom bow (Fig. 137).

FIG. 137.



Generally, the above equations, &c., are true for a negative as well as for a positive value of f.

In all cases, therefore, when the load is uniformly distributed

each point was equal to half the uniform load on the adjoining intervals those points remained on the parabola.

P and p can be found from the equations;

$$\frac{f}{F} = \frac{p}{P}, \text{ and } P - p = K.$$

Putting $\frac{f}{F} = \frac{1}{n}$ and reducing,

$$p = \frac{k}{n-1}, \; P = k + \frac{k}{n-1}.$$

The load k produces therefore a tension $\frac{k}{n-1}$ in the verticals.

In the preceding numerical example

$$\frac{1}{n} = \frac{f}{F} = \frac{1}{2 \cdot 5} = \frac{2}{5} \therefore n = \frac{5}{2}.$$

Hence the tension in the verticals (or the negative load on the lower bow) is in this case

$$p = \frac{k}{\frac{5}{2} - 1} = \tfrac{2}{3} k,$$

and the load on the upper bow is

$$P = k + p = \tfrac{5}{3} . k.$$

Thus, if the external load on each top joint is $7 \cdot 5$ tons, the tension in each vertical will be $\frac{2}{3} \times 7 \cdot 5 = 5$ tons, and the upper bow is in the same condition as if loaded with $\frac{5}{3} \times 7 \cdot 5 = 12 \cdot 5$ tons at each joint.

The load of $1 \cdot 5$ tons on each top joint due to the weight of the truss itself produces a tension in the verticals $= \frac{2}{3} \times 1.5 = 1$ ton, and the positive load on the upper bow is $\frac{5}{3} \times 1 \cdot 5 = 2 \cdot 5$ tons.

Lastly, if $7 \cdot 5 + 1 \cdot 5 = 9$ tons is the total load on each top joint, the negative load per unit of length of span on the lower bow is 6 tons, the positive load on the upper bow is 15 tons, and the tension in each vertical is 6 tons.

If, however, part of the load is applied at the lower joints it must be conveyed by secondary verticals to the top joints, and the tension in these secondary verticals is to be added to that in the main verticals. For instance, in the preceding example

0·5 ton of the weight of the truss was considered as acting on the lower joints; 0·5 ton must therefore be added to the 6 tons tension found above, and this coincides exactly with the value obtained by the method of moments. The negative load on the lower bow remains the same as before, namely 6 tons, for the tension in the secondary verticals evidently does not affect it.

The constant horizontal stress in the bows is:

$$H + \frac{p l^2}{2f} = \frac{6 \times (6 \cdot 5)^2}{2 \times 1} = 126 \cdot 7 \text{ tons},$$

which is tension in the lower bow and compression in the upper bow. This is the same value that was obtained by the method of moments ($Z_7 = + 126 \cdot 7$ tons, and $X_7 = - 126 \cdot 7$ tons).

If $\frac{1}{n} = 0$, it follows that $p = 0$ and $P = k$; that is, if the lower bow becomes a horizontal straight line the loading of the upper bow produces no tension in the verticals.

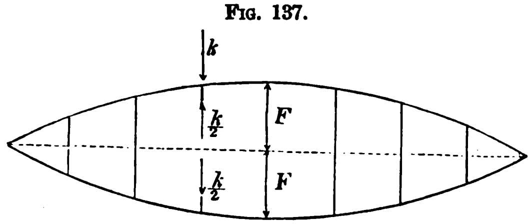
Further, if $\frac{1}{n}$ becomes negative p also becomes negative; that is, the loading of the upper bow produces compression in the verticals. For instance, if

$$\frac{1}{n} = -1,$$

$$p = -\frac{k}{2} \text{ and } P = +\frac{k}{2}.$$

In this case, therefore, one half of the load placed on the top is transferred to the bottom bow (Fig. 137).

FIG. 137.



Generally, the above equations, &c., are true for a negative as well as for a positive value of f.

In all cases, therefore, when the load is uniformly distributed

each point was equal to half the uniform load on the adjoining intervals those points remained on the parabola.

P and p can be found from the equations;

$$\frac{f}{F} = \frac{p}{P}, \text{ and } P - p = K.$$

Putting $\frac{f}{F} = \frac{1}{n}$ and reducing,

$$p = \frac{k}{n-1}, P = k + \frac{k}{n-1}.$$

The load k produces therefore a tension $\frac{k}{n-1}$ in the verticals.

In the preceding numerical example

$$\frac{1}{n} = \frac{f}{F} = \frac{1}{2 \cdot 5} = \frac{2}{5} \therefore n = \frac{5}{2}.$$

Hence the tension in the verticals (or the negative load on the lower bow) is in this case

$$p = \frac{k}{\frac{5}{2} - 1} = \tfrac{2}{3} k,$$

and the load on the upper bow is

$$P = k + p = \tfrac{5}{3} . k.$$

Thus, if the external load on each top joint is $7 \cdot 5$ tons, the tension in each vertical will be $\frac{2}{3} \times 7 \cdot 5 = 5$ tons, and the upper bow is in the same condition as if loaded with $\frac{5}{3} \times 7 \cdot 5 = 12 \cdot 5$ tons at each joint.

The load of $1 \cdot 5$ tons on each top joint due to the weight of the truss itself produces a tension in the verticals $= \frac{2}{3} \times 1.5 =$ 1 ton, and the positive load on the upper bow is $\frac{5}{3} \times 1 \cdot 5 =$ $2 \cdot 5$ tons.

Lastly, if $7 \cdot 5 + 1 \cdot 5 = 9$ tons is the total load on each top joint, the negative load per unit of length of span on the lower bow is 6 tons, the positive load on the upper bow is 15 tons, and the tension in each vertical is 6 tons.

If, however, part of the load is applied at the lower joints it must be conveyed by secondary verticals to the top joints, and the tension in these secondary verticals is to be added to that in the main verticals. For instance, in the preceding example

0·5 ton of the weight of the truss was considered as acting on the lower joints; 0·5 ton must therefore be added to the 6 tons tension found above, and this coincides exactly with the value obtained by the method of moments. The negative load on the lower bow remains the same as before, namely 6 tons, for the tension in the secondary verticals evidently does not affect it.

The constant horizontal stress in the bows is:

$$H + \frac{pl^2}{2f} = \frac{6 \times (6 \cdot 5)^2}{2 \times 1} = 126 \cdot 7 \text{ tons},$$

which is tension in the lower bow and compression in the upper bow. This is the same value that was obtained by the method of moments ($Z_7 = + 126 \cdot 7$ tons, and $X_7 = - 126 \cdot 7$ tons).

If $\frac{1}{n} = 0$, it follows that $p = 0$ and $P = k$; that is, if the lower bow becomes a horizontal straight line the loading of the upper bow produces no tension in the verticals.

Further, if $\frac{1}{n}$ becomes negative p also becomes negative; that is, the loading of the upper bow produces compression in the verticals. For instance, if

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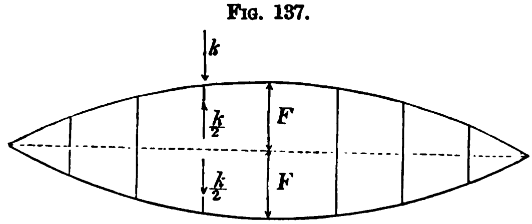
FIG. 137.



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In all cases, therefore, when the load is uniformly distributed

each point was equal to half the uniform load on the adjoining intervals those points remained on the parabola.

P and p can be found from the equations;

$$\frac{f}{F} = \frac{p}{P}, \text{and } P - p = K.$$

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$$P = k + p = \frac{5}{3} . k.$$

Thus, if the external load on each top joint is $7 \cdot 5$ tons, the tension in each vertical will be $\frac{2}{3} \times 7 \cdot 5 = 5$ tons, and the upper bow is in the same condition as if loaded with $\frac{5}{3} \times 7 \cdot 5 = 12 \cdot 5$ tons at each joint.

The load of $1 \cdot 5$ tons on each top joint due to the weight of the truss itself produces a tension in the verticals $= \frac{2}{3} \times 1.5 =$ 1 ton, and the positive load on the upper bow is $\frac{5}{3} \times 1 \cdot 5 =$ $2 \cdot 5$ tons.

Lastly, if $7 \cdot 5 + 1 \cdot 5 = 9$ tons is the total load on each top joint, the negative load per unit of length of span on the lower bow is 6 tons, the positive load on the upper bow is 15 tons, and the tension in each vertical is 6 tons.

If, however, part of the load is applied at the lower joints it must be conveyed by secondary verticals to the top joints, and the tension in these secondary verticals is to be added to that in the main verticals. For instance, in the preceding example

0·5 ton of the weight of the truss was considered as acting on the lower joints; 0·5 ton must therefore be added to the 6 tons tension found above, and this coincides exactly with the value obtained by the method of moments. The negative load on the lower bow remains the same as before, namely 6 tons, for the tension in the secondary verticals evidently does not affect it.

The constant horizontal stress in the bows is:

$$H + \frac{pl^2}{2f} = \frac{6 \times (6\cdot5)^2}{2 \times 1} = 126\cdot7 \text{ tons,}$$

which is tension in the lower bow and compression in the upper bow. This is the same value that was obtained by the method of moments ($Z_7 = +126\cdot7$ tons, and $X_7 = -126\cdot7$ tons).

If $\frac{1}{n} = 0$, it follows that $p = 0$ and $P = k$; that is, if the lower bow becomes a horizontal straight line the loading of the upper bow produces no tension in the verticals.

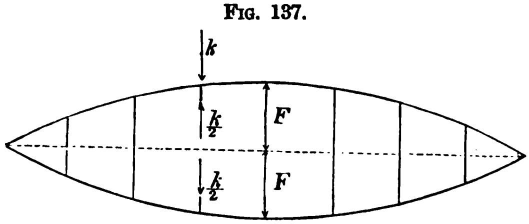
Further, if $\frac{1}{n}$ becomes negative p also becomes negative; that is, the loading of the upper bow produces compression in the verticals. For instance, if

$$\frac{1}{n} = -1,$$

$$p = -\frac{k}{2} \text{ and } P = +\frac{k}{2}.$$

In this case, therefore, one half of the load placed on the top is transferred to the bottom bow (Fig. 137).

FIG. 137.



Generally, the above equations, &c., are true for a negative as well as for a positive value of f.

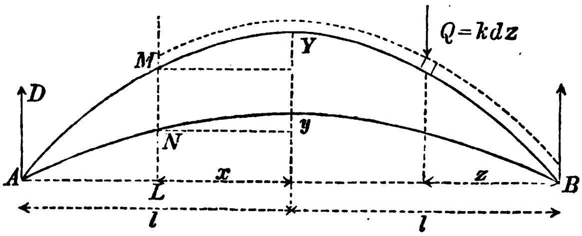
In all cases, therefore, when the load is uniformly distributed

over the span the verticals alone are capable of maintaining equilibrium. It is only when the load is unevenly distributed that there is any tendency to deformation, and this is met by the introduction of diagonals.

The law upon which the stresses in the diagonals of a double bowstring truss depend can also be found, and is appended here for those readers who are acquainted with the Calculus.

The diagonals together with the verticals make the truss perfectly rigid, and it therefore behaves towards external forces in the same manner as a simple beam

FIG. 138.

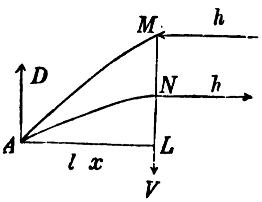


supported at both ends. Thus, if a load Q be placed on it at a horizontal distance z, from the right abutment (Fig. 138) a reaction

$$D = Q \cdot \frac{z}{2l}$$

will be produced at the other abutment A.

FIG. 139.



Take a vertical section M N through the truss to the left of the weight Q, dividing the structure into two parts, one of which is shown in Fig. 139. To maintain equilibrium forces must be applied to this section. In order that the algebraic sum of the vertical forces may be zero, a vertical force V must be applied equal to D, therefore

$$V = Q \cdot \frac{z}{2l}.$$

The force V alone would, however, with D produce a couple, and it is therefore necessary for equilibrium to apply at the section a couple of equal moment. The horizontal forces $h\ h$ form such a couple. If the section line M N is indefinitely near to one of the vertical braces, M and N are the only points at which a bar can be intersected, and the horizontal forces must therefore be applied at these points. The value of h can be found by taking moments about A, thus:

$$0 = V(l - x) - h \cdot ML + h \cdot NL.$$

If, as before, the heights of the arc of the parabolas are F and f respectively, the equations to these curves are,

$$\frac{Y}{F} = \frac{x^2}{l^2} = \frac{y}{f}.$$

From which the following values for M L and N L are obtained:

$$\mathrm{M\,L} = \mathrm{F} - \mathrm{Y} = \mathrm{F}\left(1 - \frac{x^2}{l^2}\right),$$
$$\mathrm{N\,L} = f - y = f\left(1 - \frac{x^2}{l^2}\right).$$

Substituting these, as well as the value found above for V, in the equation of moments;

$$0 = \mathrm{Q}\frac{z}{2\,l}(l - x) - h\,\mathrm{F}\left(1 - \frac{x^2}{l^2}\right) + h\,f\left(1 - \frac{x^2}{l^2}\right),$$

whence

$$h = \frac{\mathrm{Q}\,l\,z}{2\,(\mathrm{F} - f)\,(l + x)}.$$

And differentiating with respect to x,

$$\frac{d\,h}{d\,x} = -\frac{\mathrm{Q}\,l\,z}{2\,(\mathrm{F} - f)\,(l + x)^2}.$$

This differential equation gives the rate of increase of h for an increase of the abscissa x, that is when the point M moves towards the left.

The absolute value of h will evidently be greatest when the whole of the bow from B to M is loaded with weights Q. Replacing Q by $k\,d\,z$ and writing $\frac{d\,\mathrm{H}}{d\,x}$ for $\frac{d\,h}{d\,x}$ according to the previous notation used for a distributed load:

$$\frac{d\,\mathrm{H}}{d\,x} = -\frac{k\,l}{2\,(\mathrm{F} - f)\,(l + x)^2}\cdot\int_{z=0}^{z=l+x} z\,d\,z$$

$$\frac{d\,\mathrm{H}}{d\,x} = -\frac{k\,l}{2\,(\mathrm{F} - f)\,(l + x)^2}\times\frac{(l + x)^2}{2}$$

$$\therefore\frac{d\,\mathrm{H}}{d\,x} = -\frac{k\,l}{4\,(\mathrm{F} - f)}.$$

If the loading were continued to the left of the point M, this negative value of $\frac{d\,\mathrm{H}}{d\,x}$ would approach 0, and to prove this it is only necessary to ascertain as before the effect of a single load placed to the left of M, at a distance z from the abutment A.

It is then found that all such loads make $\frac{d\,h}{d\,x}$ positive, and therefore $\frac{d\,\mathrm{H}}{d\,x}$ is greatest when every point from A to M is loaded. The equations thus obtained are:

$$0 = -\,\mathrm{Q}\frac{z}{2\,l}(l + x) + h\,\mathrm{F}\left(1 - \frac{x^2}{l^2}\right) - h\,f\left(1 - \frac{x^2}{l^2}\right),$$

$$h = \frac{\mathrm{Q}\,l\,z}{2\,(\mathrm{F} - f)\,(l - x)},\qquad \frac{d\,h}{d\,x} = \frac{\mathrm{Q}\,l\,z}{2\,(\mathrm{F} - f)\,(l - x)^2},$$

$$\frac{d\,\mathrm{H}}{d\,x} = \frac{k\,l}{2\,(\mathrm{F} - f)\,(l - x)^2}\cdot\int_{z=0}^{z=l-x} z\,d\,z$$

$$= \frac{k\,l}{2\,(\mathrm{F} - f)\,(l - x)^2}\times\frac{(l - x)^2}{2}.$$

$$\frac{d\,\mathrm{H}}{d\,x} = -\frac{k\,l}{4\,(\mathrm{F}-f)},$$

The general equation for $\frac{d\,\mathrm{H}}{d\,x}$ can therefore be put in the form:

$$\frac{d\,\mathrm{H}}{d\,x}\begin{cases}\text{maximum} = +\dfrac{k\,l}{4\,(\mathrm{F}-f)}\\ \text{minimum} = -\dfrac{k\,l}{4\,(\mathrm{F}-f)}.\end{cases}$$

These results can be employed in the following manner to determine the stresses in the diagonals.

It will be remembered that the section line M N was taken indefinitely near to a vertical. The point where the diagonal is cut will therefore be at the intersection with one of the booms, for instance, the lower one (Fig. 140). The three forces H, V, H distribute themselves as follows; at the point of intersection M the force H is applied together with as much of the vertical force V as is necessary to produce a resultant in the direction of the bow; at the point of intersection

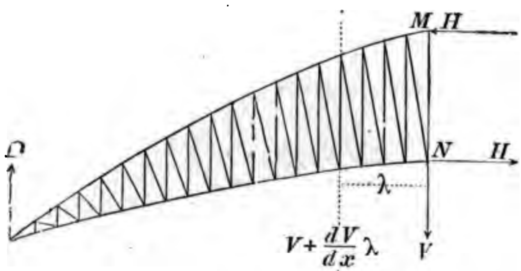


FIG. 140.

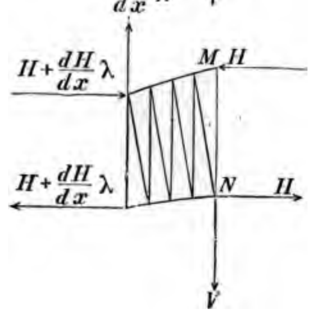


FIG. 141.

N with the diagonal a part of the horizontal force H, and as much of the vertical force V as is necessary to give a resultant in the direction of the diagonal; and lastly at the point of intersection N with the lower bow the remainder of H, and as much of the vertical force V as is necessary to produce a resultant in the direction of the lower bow.

Had the section been taken at a distance $d\,x$ further to the left, H and V would have been replaced by

$$\mathrm{H} + \frac{d\,\mathrm{H}}{d\,x}\,.\,d\,x \text{ and } \mathrm{V} + \frac{d\,\mathrm{V}}{d\,x}\,.\,d\,x$$

If both sections be taken simultaneously (Fig. 141) and the forces acting on the part of the structure thus cut out be considered it will be observed that the excess $\frac{dH}{dx} \cdot dx$ of the horizontal forces is the force that tends to move the upper bow to the right and the lower bow to the left.

If the breadth of the piece cut out be taken equal to the small quantity λ (instead of the indefinitely small quantity dx) $\frac{dH}{dx} \cdot \lambda$ will still very nearly represent the distorting force, or substituting for $\frac{dH}{dx}$ the value found above it will be very nearly equal to

$$\pm \frac{kl}{4(F - f)} \cdot \lambda$$

when a maximum or a minimum respectively.

This force distributes itself on the apices of the triangles formed by the diagonals and verticals, and acts towards the left or towards the right according to the position of the load.

FIG. 142.

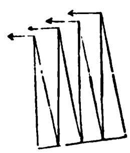
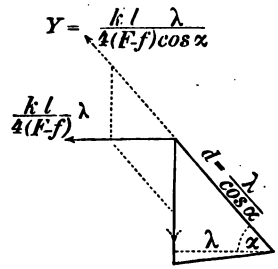


FIG. 143.



If λ is the length of one bay, the distorting force is applied to one apex only and can be resolved into two components, one along the diagonal and the other along the vertical (Fig. 143).

The component along the diagonal is

$$Y\,(\text{max. or min.}) = \pm \frac{kl}{4(F - f)} \frac{\lambda}{\cos\alpha}.$$

or since $\frac{\lambda}{\cos\alpha} = d$, the length of the diagonal,

$$Y\,(\text{max. or min.}) = \pm \frac{kl}{4(F - f)} \cdot d.$$

Therefore to find the greatest stress in any diagonal it is only necessary to multiply its length by $\pm \frac{kl}{4(F - f)}$.

For instance in the case of the roof calculated in § 15,

$$\frac{kl}{4(F - f)} = \frac{7{\cdot}5 \times 6{\cdot}5}{4(2{\cdot}5 - 1)} = 8{\cdot}125.$$

By measuring the lengths $d_2, d_3 \ldots d_{12}$ of the diagonals and multiplying by 8·125 the following table is obtained:

$$
\begin{array}{ll}
d_2 = 1\cdot018, & Y_2 = 8\cdot125 \times 1\cdot018 = 8\cdot3 \\
d_3 = 1\cdot163, & Y_3 = 8\cdot125 \times 1\cdot163 = 9\cdot5 \\
d_4 = 1\cdot361, & Y_4 = 8\cdot125 \times 1\cdot361 = 11\cdot1 \\
d_5 = 1\cdot55, & Y_5 = 8\cdot125 \times 1\cdot55 = 12\cdot6 \\
d_6 = 1\cdot7, & Y_6 = 8\cdot125 \times 1\cdot7 = 13\cdot8 \\
d_7 = 1\cdot8, & Y_7 = 8\cdot125 \times 1\cdot8 = 14\cdot6 \\
d_8 = 1\cdot835, & Y_8 = 8\cdot125 \times 1\cdot835 = 14\cdot9 \\
d_9 = 1\cdot815, & Y_9 = 8\cdot125 \times 1\cdot815 = 14\cdot7 \\
d_{10} = 1\cdot735, & Y_{10} = 8\cdot125 \times 1\cdot735 = 14\cdot1 \\
d_{11} = 1\cdot605, & Y_{11} = 8\cdot125 \times 1\cdot605 = 13\cdot0 \\
d_{12} = 1\cdot426, & Y_{12} = 8\cdot125 \times 1\cdot426 = 11\cdot6.
\end{array}
$$

Comparing these values with those given in Fig. 120 it will be seen that the differences are very small.

The above law can be applied to the case of fish-bellied girders, by writing $-f$ for f;

$$\frac{dH}{dx} = \pm \frac{kl}{4(F+f)}.$$

It is also true in the special cases when the lower or the upper bow become straight; in the first case $f = 0$ and in the second $F = 0$, or $\frac{dH}{dx} = \frac{kl}{4F}$, when the lower bow is straight, and $\frac{dH}{dx} = \frac{kl}{4f}$ when the upper bow is straight.

For instance in the parabolic girder calculated in § 6,

$$\frac{dH}{dx} = \frac{kl}{4f} = \frac{2500 \times 8}{4 \times 2} = 2500,$$

and measuring the lengths $d_2, d_3 \ldots d_7$ of the diagonals:

$$
\begin{array}{ll}
d_2 = 2\cdot5 & Y_2 = 2500 \times 2\cdot5 = 6250 \\
d_3 = 2\cdot741 & Y_3 = 2500 \times 2\cdot741 = 6850 \\
d_4 = 2\cdot828 & Y_4 = 2500 \times 2\cdot828 = 7070 \\
d_5 = 2\cdot741 & Y_5 = 2500 \times 2\cdot741 = 6850 \\
d_6 = 2\cdot5 & Y_6 = 2500 \times 2\cdot5 = 6250 \\
d_7 = 2\cdot183 & Y_7 = 2500 \times 2\cdot183 = 5460
\end{array}
$$

These stresses agree almost exactly with those given in Fig. 27.

It is possible to investigate a similar law for the stresses in the verticals, but on account of their double function, first as braces and secondly as struts or ties to convey the load from one joint to another, this law is very complicated and consequently unsuited to practical purposes. Nor would the results agree with those obtained by the method of moments as well as in the case of the diagonals. For these general laws are based upon the supposition that the moving load progresses gradually, whereas when using the method of moments it is considered that the moving load advances by jumps from one joint to the next. It is therefore better in all cases to employ the method of moments to calculate the stresses in the verticals.

FIFTH CHAPTER.

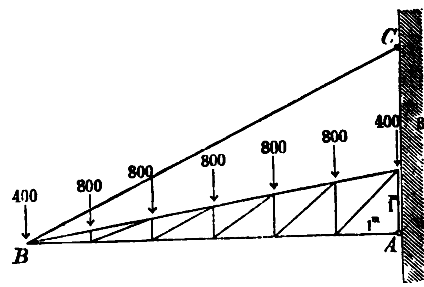
§ 19.—Cantilever Roof with Stay, Span 6 Metres.

The load, including snow and wind, is assumed to be 200 kilos. per square metre of horizontal area, covered. The distance apart of the principals is 4 metres. The load on each principal is therefore

$$6 \times 4 \times 200 = 4800 \text{ kilos.}$$

and the load on each of the 6 bays is 800 kilos.; of the 7 joints, the first and the last have 400 kilos. to support, and

Fig. 144.



the remaining five 800 kilos. (Fig. 144). The weight of the truss itself being small, the whole of this load may be taken as a variable load.*

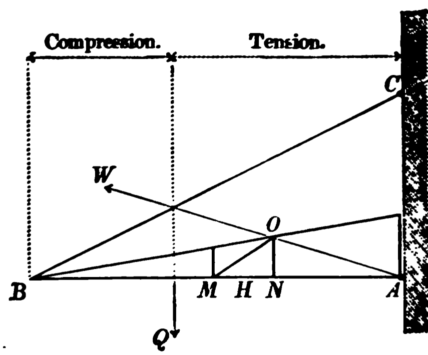
Calculation of the Stress H in the Horizontal Bars.

The reactions W and P produced at the two points of support A and C by a load Q are shown in Fig. 145. To find the stress H in the bar M N, due to this load, the equation of moments about the point O, for the part of

* As will be seen in the sequel, Prof. Ritter understands by the load being variable that any joint or joints may be loaded and the rest unloaded. This, it will be observed, is not the usual English practice in the case of roofs.—Trans.

the roof given in Fig. 146, would have to be formed. But the position of Q has been so chosen that the resultant of Q and P passes through O, and consequently H = *o*. It will also easily be seen that all loads to the left of Q produce negative stresses, and all loads to the right of Q positive stresses, in the bar M N. Hence, when H is a minimum, the part of the roof over

FIG. 145.



which "Compression" is written in Fig. 145 will be loaded, and the remainder unloaded; and when H is a maximum, the loads will extend over the part marked "Tension."

FIG. 146.

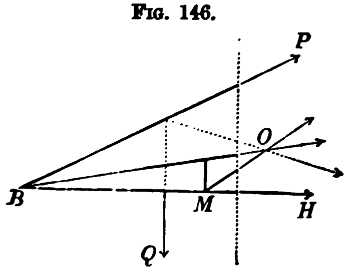
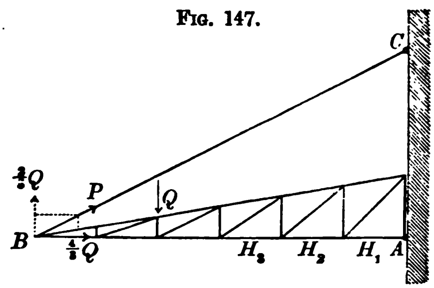


FIG. 147.



The same result can be arrived at, however, by forming the equation of moments, when all the joints are loaded, and arranging this equation so that the effect of every load can be seen.

When Q (distant 4 metres from the wall) is the only load on the roof, a stress P is produced in the rod B C, whose vertical component is $\frac{2}{3}$ Q (Fig. 147); for the equation

of moments about A shows that the vertical component of P acts in the same manner as the reaction of the point of support B would if A B were a girder resting on two supports, A and B. Since A B is to A C as 6 to 3, it follows that the horizontal component of P is always twice as great as the vertical component, and is in this case therefore equal to $\frac{4}{3}$ Q. Thus, the equation of moments to find H_3 is (Fig. 148)—

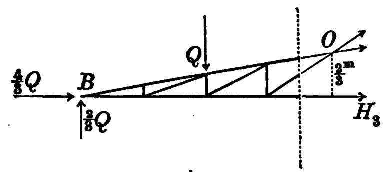
$$0 = -H_3 \times \tfrac{2}{3} - Q \times 2 + \tfrac{2}{3} Q \times 4 - \tfrac{4}{3} Q \times \tfrac{2}{3},$$

or

$$H_3 \times \tfrac{2}{3} = -Q \{2 - \tfrac{2}{3} . 4 + \tfrac{4}{3} . \tfrac{2}{3}\}.$$

The increment to the stress H_3 produced by Q is therefore composed of three parts. The first is the direct influence of the load, and the other two the indirect effect produced by the reactions.

FIG. 148.



If Q, however, were situated to the right of the section line, the increment to the stress would be composed of two terms only, both the indirect effect of the reactions. For instance, the increment to the stress H_3, produced by a load Q, 2 metres from the wall, is to be found from the equation

$$0 = -H_3 \times \tfrac{2}{3} + \tfrac{1}{3} Q \times 4 - \tfrac{2}{3} Q \times \tfrac{2}{3},$$

or

$$H_3 \times \tfrac{2}{3} = Q \ \{\tfrac{1}{3} . 4 - \tfrac{2}{3} . \tfrac{2}{3}\}.$$

Thus the equation to find H_3, when all the joints are loaded, is

$$\begin{aligned} H_3 \times \tfrac{2}{3} = 800 \, (\tfrac{1}{6} . 4 - \tfrac{1}{3} . \tfrac{2}{3}) + 800 \, (\tfrac{1}{3} . 4 - \tfrac{2}{3} . \tfrac{2}{3}) \\ + 800 \, (\tfrac{1}{2} . 4 - 1 . \tfrac{2}{3} - 1) - 800 \, (2 - \tfrac{2}{3} . 4 + \tfrac{4}{3} . \tfrac{2}{3}) \\ + 800 \, (3 - \tfrac{5}{6} . 4 + \tfrac{5}{3} . \tfrac{2}{3}) - 400 \, (4 - 1 . 4 + 2 . \tfrac{2}{3}). \end{aligned}$$

Omitting the negative members from the right-hand side of the equation,

$$H_3 \text{ (max.)} = + 2000 \text{ kilos.};$$

and leaving out the positive members,

$$H_3 \text{ (min.)} = - 2000 \text{ kilos.}$$

The following equations for the remaining horizontal bars are obtained in a similar manner:

$$\begin{aligned}H_1 \times 1 = &- 800\left(1 - \tfrac{1}{6} \cdot 6 + \tfrac{1}{3} \cdot 1\right) - 800\left(2 - \tfrac{1}{3} \cdot 6 + \tfrac{2}{3} \cdot 1\right)\\ &- 800\left(3 - \tfrac{1}{2} \cdot 6 + 1 \cdot 1\right) - 800\left(4 - \tfrac{2}{3} \cdot 6 + \tfrac{4}{3} \cdot 1\right)\\ &- 800\left(5 - \tfrac{5}{6} \cdot 6 + \tfrac{5}{3} \cdot 1\right) - 400\left(6 - 1 \cdot 6 + 2 \cdot 1\right)\end{aligned}$$

$$H_1\ (\text{max.}) = 0, \quad H_1\ (\text{min.}) = -4800 \text{ kilos.}$$

$$\begin{aligned}H_2 \times \tfrac{5}{6} = &\ 800\left(\tfrac{1}{6} \cdot 5 - \tfrac{1}{3} \cdot \tfrac{5}{6}\right) + 800\left(\tfrac{1}{3} \cdot 5 - \tfrac{2}{3} \cdot \tfrac{5}{6} - 1\right)\\ &- 800\left(2 - \tfrac{1}{2} \cdot 5 + 1 \cdot \tfrac{5}{6}\right) - 800\left(3 - \tfrac{2}{3} \cdot 5 + \tfrac{4}{3} \cdot \tfrac{5}{6}\right)\\ &- 800\left(4 - \tfrac{5}{6} \cdot 5 + \tfrac{5}{3} \cdot \tfrac{5}{6}\right) - 400\left(5 - 1 \cdot 5 + 2 \cdot \tfrac{5}{6}\right)\end{aligned}$$

$$H_2\ (\text{max.}) = +640 \text{ kilos.}, \quad H_2\ (\text{min.}) = -3040 \text{ kilos.}$$

$$\begin{aligned}H_4 \times \tfrac{1}{2} = &\ 800\left(\tfrac{1}{6} \cdot 3 - \tfrac{1}{3} \cdot \tfrac{1}{2}\right) + 800\left(\tfrac{1}{3} \cdot 3 - \tfrac{2}{3} \cdot \tfrac{1}{2}\right)\\ &+ 800\left(\tfrac{1}{2} \cdot 3 - 1 \cdot \tfrac{1}{2}\right) + 800\left(\tfrac{2}{3} \cdot 3 - \tfrac{4}{3} \cdot \tfrac{1}{2} - 1\right)\\ &- 800\left(2 - \tfrac{5}{6} \cdot 3 + \tfrac{5}{3} \cdot \tfrac{1}{2}\right) - 400\left(3 - 1 \cdot 3 + 2 \cdot \tfrac{1}{2}\right)\end{aligned}$$

$$H_4\ (\text{max.}) = +3733 \text{ kilos.}, \quad H_4\ (\text{min.}) = -1333 \text{ kilos.}$$

$$\begin{aligned}H_5 \times \tfrac{1}{3} = &\ 800\left(\tfrac{1}{6} \cdot 2 - \tfrac{1}{3} \cdot \tfrac{1}{3}\right) + 800\left(\tfrac{1}{3} \cdot 2 - \tfrac{2}{3} \cdot \tfrac{1}{3}\right)\\ &+ 800\left(\tfrac{1}{2} \cdot 2 - 1 \cdot \tfrac{1}{3}\right) + 800\left(\tfrac{2}{3} \cdot 2 - \tfrac{4}{3} \cdot \tfrac{1}{3}\right)\\ &+ 800\left(\tfrac{5}{6} \cdot 2 - \tfrac{5}{3} \cdot \tfrac{1}{3} - 1\right) - 400\left(2 - 1 \cdot 2 + 2 \cdot \tfrac{1}{3}\right)\end{aligned}$$

$$H_5\ (\text{max.}) = +5600 \text{ kilos.}, \quad H_5\ (\text{min.}) = -800 \text{ kilos.}$$

$$\begin{aligned}H_6 \times \tfrac{1}{6} = &\ 800\left(\tfrac{1}{6} \cdot 1 - \tfrac{1}{3} \cdot \tfrac{1}{6}\right) + 800\left(\tfrac{1}{3} \cdot 1 - \tfrac{2}{3} \cdot \tfrac{1}{6}\right)\\ &+ 800\left(\tfrac{1}{2} \cdot 1 - 1 \cdot \tfrac{1}{6}\right) + 800\left(\tfrac{2}{3} \cdot 1 - \tfrac{4}{3} \cdot \tfrac{1}{6}\right)\\ &+ 800\left(\tfrac{5}{6} \cdot 1 - \tfrac{5}{3} \cdot \tfrac{1}{6}\right) - 400\left(1 - 1 \cdot 1 + 2 \cdot \tfrac{1}{6}\right)\end{aligned}$$

$$H_6\ (\text{max.}) = +8000 \text{ kilos.}, \quad H_6\ (\text{min.}) = -800 \text{ kilos.}$$

For all the remaining bars the turning point lies in the line A B; and since the resultant W, of any load Q and the tension P produced by it in B C, always passes through A, it follows that the greatest stress in all the remaining bars occurs when every joint is loaded.

This total load of 4800 kilos. can be considered to act at the centre of A B, and the vertical component of P will then be $\frac{1}{2} \cdot 4800 = 2400$ kilos. The horizontal component of P is twice as great, or 4800 kilos. Consequently,

$$P = \sqrt{2400^2 + 4800^2} = 5367 \text{ kilos.},$$

and this is the greatest tension in B C. From Figs. 149 and 150 the lever-arm of the stress X_3, with respect to the point M, is

$$LM \cdot \cos \alpha = \tfrac{1}{2} \cdot \frac{6}{\sqrt{6^2 + 1^2}} = 0{\cdot}4932 \text{ metre.}$$

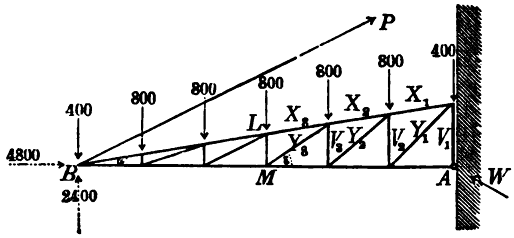
The equation of moments to determine X_3 is therefore (Fig. 150)

$$0 = X_3 \times 0{\cdot}4932 + 2400 \times 3 - 800\,(0 + 1 + 2 + \tfrac{3}{2}),$$

or

$$X_3 = -\ 7299 \text{ kilos.}$$

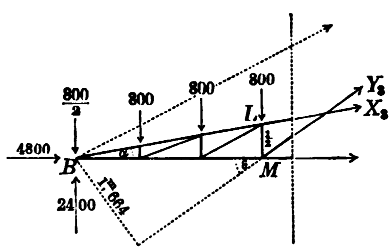
FIG. 149.



Similarly,

$$\begin{aligned}
0 &= X_1 \times 0{\cdot}822 + 2400 \times 5 - 800\,\{1 + 2 + 3 + 4 + \tfrac{5}{2}\} \\
X_1 &= -\ 2433 \text{ kilos.} \\
0 &= X_2 \times 0{\cdot}6576 + 2400 \times 4 - 800\,\{1 + 2 + 3 + \tfrac{4}{2}\} \\
X_2 &= -\ 4866 \text{ kilos.} \\
0 &= X_4 \times 0{\cdot}3288 + 2400 \times 2 - 800\,(1 + \tfrac{2}{2}) \\
X_4 &= -\ 9732 \text{ kilos.} \\
0 &= X_5 \times 0{\cdot}1644 + 2400 \times 1 - 400 \times 1 \\
X_5 &= -\ 12166 \text{ kilos.} \\
0 &= X_6 \times 0{\cdot}1644 + 2400 \times 1 - 400 \times 1 \\
X_6 &= -\ 12166 \text{ kilos.}
\end{aligned}$$

FIG. 150.

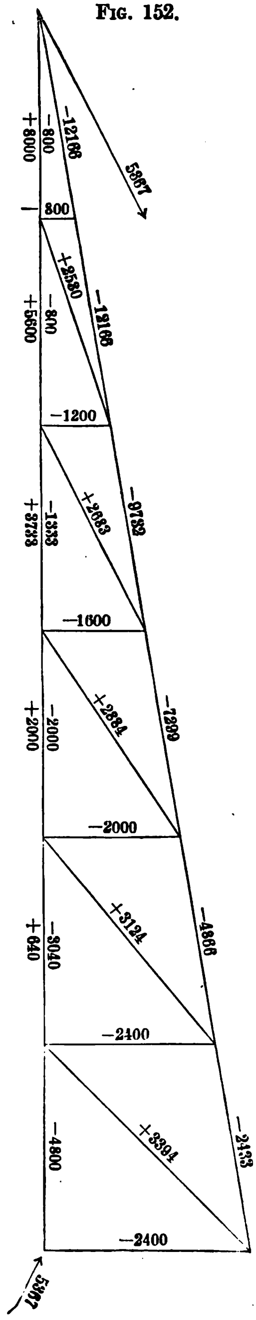


To find the stresses in the diagonals, moments will have to be taken about the point B. The lever-arm of Y_3 (Fig. 150) with respect to this point is

$$B\,M \,.\, \sin \epsilon = 3 \,.\, \frac{\tfrac{2}{3}^{*}}{\sqrt{1^2 + (\tfrac{2}{3})^2}} = 1{\cdot}664 \text{ metre.}$$

* The length of V_3 (Fig. 149) is evidently $\frac{2}{3}$ metre.—TRANS.

FIG. 152.



and the equation of moments

$$0 = -Y_3 \times 1{\cdot}664 + 800\,(1 + 2 + 3),$$

or

$$Y_3 = +\ 2884 \text{ kilos.}$$

Similarly,

$$0 = -Y_1 \times 3{\cdot}536 + 800\,(1 + 2 + 3 + 4 + 5)$$
$$Y_1 = +\ 3394 \text{ kilos.}$$

$$0 = -Y_2 \times 2{\cdot}561 + 800\,(1 + 2 + 3 + 4)$$
$$Y_2 = +\ 3124 \text{ kilos.}$$

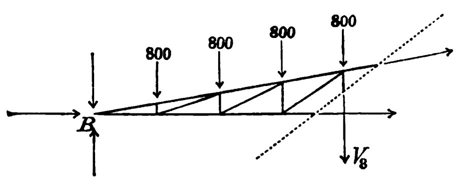
$$0 = -Y_4 \times 0{\cdot}89 + 800\,(1 + 2)$$
$$Y_4 = +\ 2683 \text{ kilos.}$$

$$0 = -Y_5 \times 0{\cdot}316 + 800 \times 1$$
$$Y_5 = +\ 2530 \text{ kilos.}$$

The stress in the verticals is also to be found by taking moments about B. Thus, for V_3 the equation of moments is (Fig. 151)

$$0 = V_3 \times 4 + 800\,(4 + 3 + 2 + 1),$$
$$V_3 = -\ 2000 \text{ kilos.}$$

FIG. 151.



Similarly,

$$0 = V_1 \times 6 + 800\,(\tfrac{6}{2} + 5 + 4 + 3 + 2 + 1)$$
$$V_1 = -\ 2400 \text{ kilos.}$$

$$0 = V_2 \times 5 + 800\,(5 + 4 + 3 + 2 + 1)$$
$$V_2 = -\ 2400 \text{ kilos.}$$

$$0 = V_4 \times 3 + 800\,(3 + 2 + 1)$$
$$V_4 = -\ 1600 \text{ kilos.}$$

$$0 = V_5 \times 2 + 800\,(2 + 1)$$
$$V_5 = -\ 1200 \text{ kilos.}$$

$$0 = V_6 \times 1 + 800 \times 1$$
$$V_6 = -\ 800 \text{ kilos.}$$

The reaction W of the fixed point of support A can be found from its components H_1 and V_1, and its greatest value is,

$$W = \sqrt{H_1^2 + V_1^2} = \sqrt{4800^2 + 2400^2} = 5367 \text{ kilos.}$$

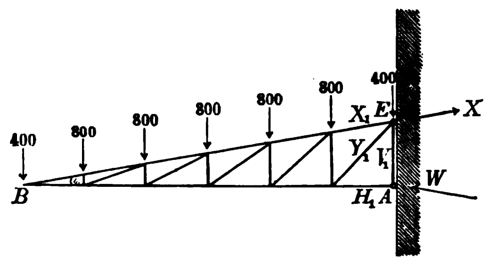
or numerically the same as the tension in B C.

The stresses obtained are collected together in Fig. 152.

§ 20.—Cantilever Roof without Stay.

The dimensions and loading of the roof are shown in Fig. 153, and are the same as those of the similar roof given in Fig. 144. Instead, however, of the toe being supported by a

Fig. 153.



stay, as in the former case, the roof is tied into the wall at the point E.

The lever-arm with respect to A of the stress X, in the bar tying the roof into the wall at E, is

$$A\,E\,.\,\cos\alpha = 1\,.\,\frac{6}{\sqrt{6^2+1^2}} = 0{\cdot}9864.$$

And hence the equation of moments to determine X is

$$0 = X \times 0{\cdot}9864 - 800\,(1 + 2 + 3 + 4 + 5 + \tfrac{6}{2}),$$

or

$$X = +\,14599 \text{ kilos.}$$

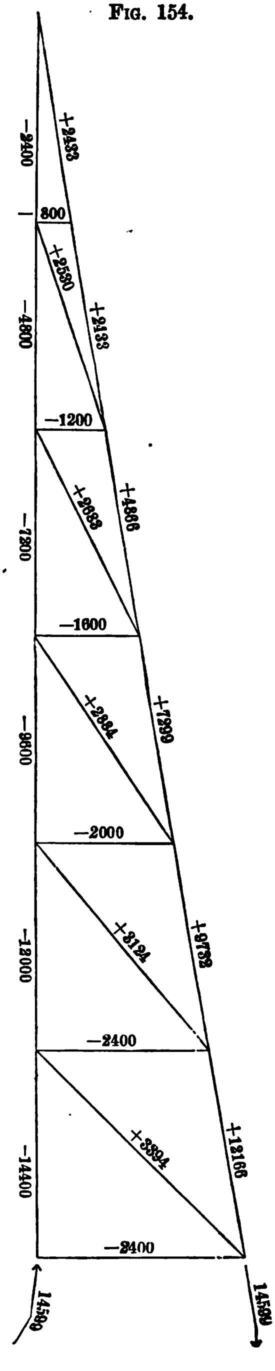


FIG. 154.

To find $X_1 \ldots X_6$, the corresponding equations in the last paragraph can be employed, by putting P and its components = 0. Thus the following equations are obtained:—

$$0 = X_1 \times 0{\cdot}822 - 800\,(1 + 2 + 3 + 4 + \tfrac{5}{2})$$
$$X_1 = +\ 12166 \text{ kilos.}$$
$$0 = X_2 \times 0{\cdot}6576 - 800\,(1 + 2 + 3 + \tfrac{4}{2})$$
$$X_2 = +\ 9732 \text{ kilos.}$$
$$0 = X_3 \times 0{\cdot}4932 - 800\,(1 + 2 + \tfrac{3}{2})$$
$$X_3 = +\ 7299 \text{ kilos.}$$
$$0 = X_4 \times 0{\cdot}3288 - 800\,(1 + \tfrac{2}{2})$$
$$X_4 = +\ 4866 \text{ kilos.}$$
$$0 = X_5 \times 0{\cdot}1644 - 400 \times 1$$
$$X_5 = +\ 2433 \text{ kilos.}$$
$$0 = X_6 \times 0{\cdot}1644 - 400 \times 1$$
$$X_6 = +\ 2433 \text{ kilos.}$$

The same applies to the stresses $H_1 \ldots H_6$, and the following equations are deduced from the former ones by putting P and its components = 0.

$$0 = -H_1 \times 1 - 800\,(1 + 2 + 3 + 4 + 5 + \tfrac{6}{2})$$
$$H_1 = -\ 14400 \text{ kilos.}$$
$$0 = -H_2 \times \tfrac{5}{6} - 800\,(1 + 2 + 3 + 4 + \tfrac{5}{2})$$
$$H_2 = -\ 12000 \text{ kilos.}$$
$$0 = -H_3 \times \tfrac{4}{6} - 800\,(1 + 2 + 3 + \tfrac{4}{2})$$
$$H_3 = -\ 9600 \text{ kilos.}$$
$$0 = -H_4 \times \tfrac{3}{6} - 800\,(1 + 2 + \tfrac{3}{2})$$
$$H_4 = -\ 7200 \text{ kilos.}$$
$$0 = -H_5 \times \tfrac{2}{6} - 800\,(1 + \tfrac{2}{2})$$
$$H_5 = -\ 4800 \text{ kilos.}$$
$$0 = -H_6 \times \tfrac{1}{6} - 400 \times 1$$
$$H_6 = -\ 2400 \text{ kilos.}$$

The equations of moments to find the stresses $V_1 \ldots V_6$ and $Y_1 \ldots Y_6$ must be formed with reference to the point B as in the

former case. In the preceding example the force P passed through B; it had, therefore, no influence on the stresses in the diagonals and verticals. Thus the stresses found for the verticals and diagonals in the former example hold good in this.

The reaction W can be found from its components V_1 and H_1, thus:—

$$W = \sqrt{V_1^2 + H_1^2} = \sqrt{2400^2 + 14400^2} = 14599 \text{ kilos.}$$

It is, therefore, numerically equal to the tension X.

The stresses in the various bars are collected together in Fig. 154.

SIXTH CHAPTER.

§ 21.—Braced Arch of 24 Metres Span.

The bridge is designed to carry a single line of railway, and is supported by two braced arches. The moving load on the bridge is taken at 4000 kilos. per metre run, of which, therefore, one-half comes on each braced arch, and the length of a bay being 3 metres, the moving load on each joint is 6000 kilos., or 6 tons (1000 kilos. to the ton). The dead load is estimated at 1400 kilos. per metre run, or 700 kilos. for each arch; that is, 2100 kilos. on each joint, or approximately 2 tons.

The two halves of the arch are in contact at the point S only (Fig. 155), and the connection is made by means of a single bolt, thus forming a hinge.* Hinged joints are also placed at the abutments A and A_1.

Preparatory to finding the greatest stresses, the effect of a single load placed on the weightless structure will be investigated.

A load Q placed anywhere on the right half of the arch produces a reaction R at the hinge S (Fig. 156), between the two halves of the arch. For the left half, the direction of this force must pass through the point A, for otherwise rotation round this point would take place. This force produces at A a reaction R, acting in the direction A S; this must be its direction, or else rotation would ensue round S, besides which action and reaction are equal and opposite. Let P be the intersection of the two forces R and Q, then it is easy to see,

* It would be a more rational form of construction if the hinge were situated in the horizontal $B B_1$. But the above construction is more general, and in the case of wooden structures the rational form would be difficult of execution. For these reasons it has not been adopted here, but can easily be deduced by making $S C = 0$ instead of $0 \cdot 5$.

by taking moments about this point, that the reaction D produced at the hinge A_1 must pass through P, in order that equilibrium may obtain. This reaction is also evidently equal in magnitude and opposite in direction to the resultant of R and Q.

Thus, to find the direction of the reactions at the abutments due to a load Q placed on one half of the arch, the line joining the hinge at the abutment of the other half with

FIG. 155.

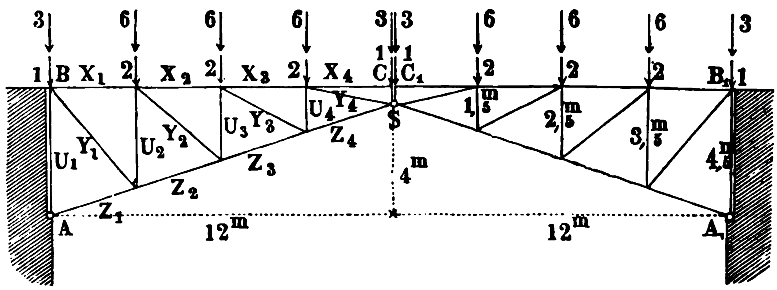
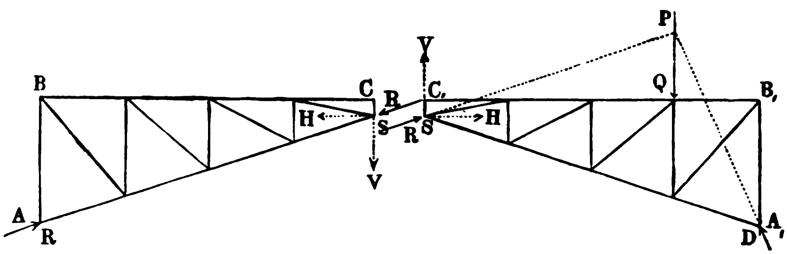


FIG. 156.



the central hinge is produced to intersect the vertical through the load, and from this point a line is drawn to the hinge at the other abutment. The pressure at the central hinge on the unloaded half is always directed to the hinge at its abutment. (In the sequel the central hinge will be called "the hinge," and the other two hinges the "abutments.")

The magnitude of the hinge-reaction R can be found by resolving it into its horizontal and vertical components, and then forming two equations of moments, one for each half

of the arch. Thus, if H and V are these components, the following equations are obtained from Fig. 156:

$$0 = V \times 12 + H \times 4 - Q \times 3,$$
$$0 = V \times 12 - H \times 4,$$

whence

$$V = \frac{Q}{8} \quad \text{and} \quad H = \frac{3\,Q}{8}.$$

Thus having found the action of a single load Q on the whole arch, it remains to determine the stresses this load produces on the various bars composing the structure. This is best done by taking a section through any three bars, as before, and writing the equation of moments for the part of the arch comprised between this section and the hinge. As in former cases, the moments are taken about the point of intersection of two of the bars cut through. Whether any particular load produces tension or compression in the bar under consideration, can easily be determined by noticing in which direction the load tends to make the part of the arch rotate. In this manner the joints that must be loaded to produce tension in a bar, and those which must be loaded to produce compression, can easily be ascertained. The maximum stress is found by loading all the former, and the minimum stress by loading the latter only.

[NOTE.—It is necessary to know the direction in which the vertical component V of the central hinge-reaction acts on each half of the arch. By examining the various figures given, it will be evident that this can always be decided on by inspection, but it would, perhaps, be safer to assume some direction as the positive one; for instance, let V be positive when it acts upwards against the left half of the arch (as in Fig. 160), then a negative value of V would indicate the state of things in Fig. 157 or Fig. 167.]

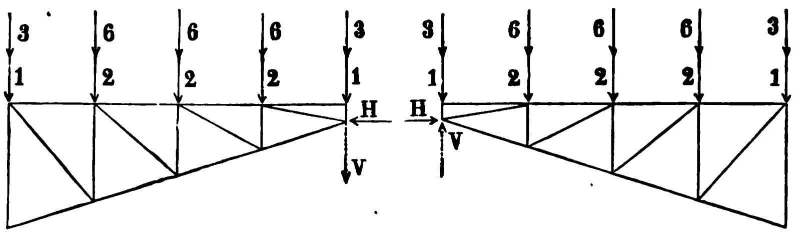
Calculation of the Stresses X in the Horizontal Bars.

The equation of moments to find X will evidently, in every case, be taken about the foot of the diagonal (Fig. 158). A load on the left half of the arch produces a hinge-reaction in

the direction A_1 S, and the resultant of this reaction and the load tends to turn the part of the arch between the section line and the hinge from right to left—that is, in the same direction as X tends to make it rotate. For equilibrium, therefore, X must be negative.

A load on the right half of the arch produces a hinge-reaction, which passes through the point round which moments

FIG. 157.



are taken; such a load will, therefore, have no effect on the stress in X. Obviously, therefore, X is always in compression. Hence, to find the greatest compression or minimum stress in X, the whole of the left half of the arch must be considered loaded, and the other half can be loaded or not, the result in either case being the same. For simplicity, both halves will be considered loaded (Fig. 157). The equations to obtain the hinge-reaction are then

$$0 = V \times 12 + H \times 4 - 4 \times 12 - 8\,(9 + 6 + 3),$$

$$0 = V \times 12 - H \times 4 + 4 \times 12 + 8\,(9 + 6 + 3),$$

whence

$$V = 0, \qquad H = 48.$$

Consequently, the equation of moments to determine X_1 (Fig. 158), with respect to the point E, is

$$0 = -X_1 \times 3{\cdot}5 - 48 \times 3 + 8\,(3 + 6) + 4 \times 9,$$

or

$$X_1 \text{ (min.)} = -10{\cdot}29 \text{ tons.}$$

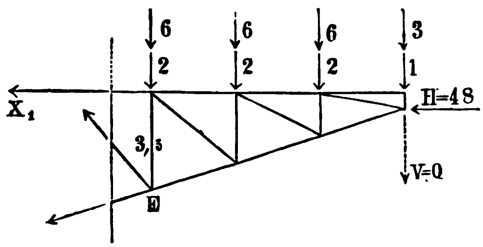
The following equations are obtained in the same manner:

$$0 = -X_2 \times 2\cdot5 - 48 \times 2 + 8 \times 3 + 4 \times 6$$
$$X_2\ (\text{min.}) = -19\cdot2 \text{ tons}$$

$$0 = -X_3 \times 1\cdot5 - 48 \times 1 + 4 \times 3$$
$$X_3\ (\text{min.}) = -24 \text{ tons}$$

$$0 = -X_4 \times 0\cdot5$$
$$X_4 = 0.$$

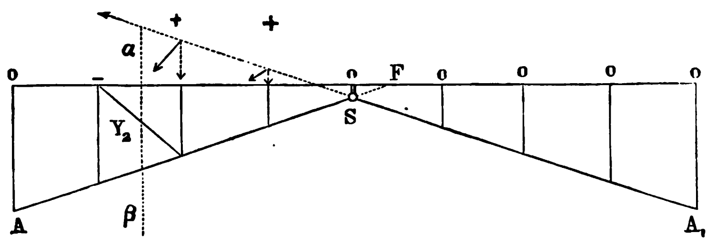
FIG. 158.



Calculation of the Stresses Y in the Diagonals.

The stress in the diagonal Y_2 will be calculated to illustrate the method. The loads can be divided into three groups. Those in the first group make Y_2 positive, those in the second

FIG. 159.



negative, and lastly, those in the third exert no influence; and therefore, if acting alone, the stress in Y_2 would be zero.

These groups are shown in Fig. 159, by the signs +, —, and 0.

The loads on the third and fourth joints belong to the first

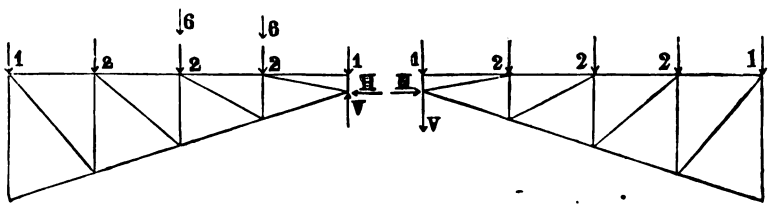
group, for the resultants of these loads and the hinge-reactions produced by them tend to turn the part of the arch between the section line $\alpha\ \beta$ and the hinge, from right to left. The stress Y_2 has the opposite tendency (Fig. 161), and is therefore made positive by these loads.

The load on the second joint is the only one belonging to the second group. This load does not act directly on the part of the arch under consideration, but by means of the hinge-reaction produced by it, which acts in the direction A_1 S, thus tending to produce rotation from left to right, or, in other words, making the stress Y_2 negative.

The third group contains the loads on all the remaining joints; for either they produce no hinge-reaction (1st and 9th), and have therefore no influence, or else they act indirectly through a hinge-reaction in the direction A S, passing through the point F, round which moments are taken, and consequently producing no stress in Y_2.

To determine Y_2 (max.), therefore, the 3rd and 4th joints are to be loaded, and the 2nd is to remain unloaded. (The

FIG. 160.



other joints may be loaded or not; they will, however, be considered as unloaded.) The hinge-reaction for this loading must now be found from the equations. (Fig. 160.)

$$0 = -V \times 12 + H \times 4 - 1 \times 12 - 2(9 + 6 + 3),$$

$$0 = -V \times 12 - H \times 4 + 1 \times 12 + 2(9 + 6 + 3) + 6(9 + 6),$$

whence

$$V = 3 \cdot 75, \qquad H = 23 \cdot 25;$$

and the equation of moments for the part shown in Fig. 161 about F is therefore

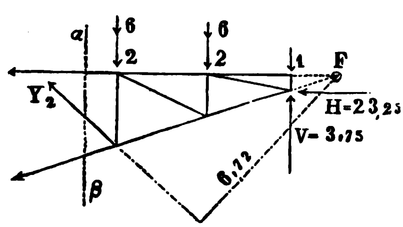
$$0 = Y_2 \times 6\cdot72 + 23\cdot25 \times 0\cdot5 + 3\cdot75 \times 1\cdot5 - 1 \times 1\cdot5 - 8\,(4\cdot5 + 7\cdot5),$$

or

$$Y_2 \text{ (max.)} = +\ 11\cdot94 \text{ tons.}$$

To determine Y_2 (min.) the 3rd and 4th joints must be unloaded, and the second joint loaded (the remaining joints will be considered unloaded). The components of the hinge-reaction can be found from the equations:

FIG. 161.



$$0 = -V \times 12 + H \times 4 - 1 \times 12 - 2\,(9 + 6 + 3),$$

$$0 = -V \times 12 - H \times 4 + 1 \times 12 + 2\,(9 + 6 + 3) + 6 \times 3,$$

$$V = 0\cdot75, \qquad H = 14\cdot25;$$

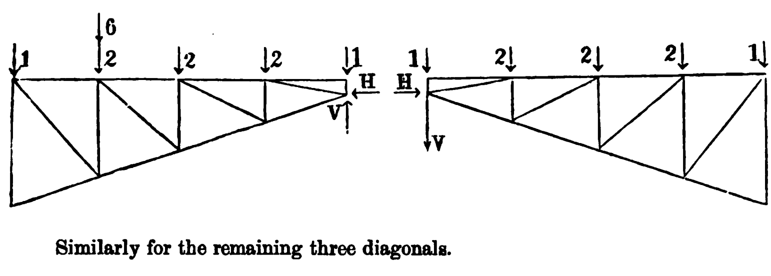
whence, from Fig. 163, the equation of moments is

$$0 = Y_2 \times 6\cdot72 + 0\cdot75 \times 1\cdot5 + 14\cdot25 \times 0\cdot5 - 1 \times 1\cdot5 - 2\,(4\cdot5 + 7\cdot5),$$

therefore,

$$Y_2 \text{ (min.)} = +\ 2\cdot57 \text{ tons.}$$

FIG. 162.



Similarly for the remaining three diagonals.

Y_1 (max.).

Take every joint loaded, then

$$V = 0, \qquad H = 48,$$

and

$$0 = Y_1 \times 10\cdot25 + 48 \times 0\cdot5 - 4 \times 1\cdot5 - 8\,(4\cdot5 + 7\cdot5 + 10\cdot5),$$

$$Y_1 \text{ (max.)} = +\ 15\cdot8 \text{ tons.}$$

Y_1 (min.) need not be considered, as no distribution of the load produces compression in this diagonal.

Y_2 (max.).

The 4th joint only is to be loaded, then

$$V = 2 \cdot 25, \qquad H = 18 \cdot 75,$$

and

$$0 = Y_2 \times 3 \cdot 35 + 2 \cdot 25 \times 1 \cdot 5 + 18 \cdot 75 \times 0 \cdot 5 - 1 \times 1 \cdot 5 - 8 \times 4 \cdot 5.$$

$$Y_2 \text{ (max.)} = + 7 \cdot 38 \text{ tons},$$

Y_2 (min.).

The 2nd and third joints only are to be loaded, then

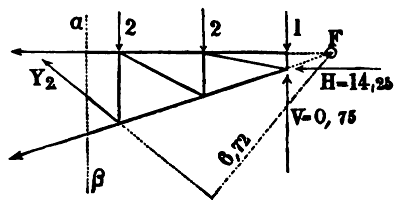
$$V = 2 \cdot 25, \quad H = 18 \cdot 75,$$

and

$$0 = Y_2 \times 3 \cdot 35 + 2 \cdot 25 \times 1 \cdot 5 + 18 \cdot 75 \times 0 \cdot 5 - 1 \times 1 \cdot 5 - 2 \times 4 \cdot 5.$$

$$Y_2 \text{ (min.)} = - 0 \cdot 67 \text{ tons},$$

FIG. 163.



Y_4 (max.).

Y_4 (max.) need not be considered, as no distribution of the load produces tension in this diagonal.

Y_4 (min.).

To obtain Y_4 (min.) joints 2, 3, and 4 are to be loaded, then

$$V = 4 \cdot 5, \qquad H = 25 \cdot 5,$$

and

$$0 = Y_4 \times 0 \cdot 738 + 4 \cdot 5 \times 1 \cdot 5 + 25 \cdot 5 \times 0 \cdot 5 - 1 \times 1 \cdot 5.$$

$$Y_4 \text{ (min.)} = - 24 \cdot 4 \text{ tons}.$$

Calculation of the Stresses Z in the Lower Bars.

The stress in the bar Z_3 will be calculated, to illustrate the method. In this case moments will be taken about the point J (Fig. 164). A vertical through G, the point of intersection of A J and A_1 S, gives the position of the load which produces no stress in Z_3, for the resultant R, of a load Q in this position and its hinge-reaction D, passes through the point J. Any load to the right of G will produce compression in Z_3, for the resultant R then passes to the right of J, and the tendency is to turn the part of the arch under consideration (Fig. 166) from left to right; Z_3 has the same tendency, and must therefore be negative to maintain equilibrium. Any load to the left of G, on the contrary, produces tension in Z_3, for the resultant

R passes to the left of J. The vertical through G is therefore what may be termed the *loading boundary* between the loads producing tension and those producing compression.

Thus (Fig. 165) the components of the hinge-reaction when Z_3 is a maximum are to be found from the equations

$$0 = -V \times 12 + H \times 4 - 1 \times 12 - 2(9 + 6 + 3),$$

$$0 = -V \times 12 - H \times 4 + 1 \times 12 + 2(9 + 6 + 3) + 6(6 + 3),$$

$$V = 2 \cdot 25, \qquad H = 18 \cdot 75;$$

whence from Fig. 166 the equation of moments is

$$0 = Z_3 \times 2 \cdot 37 - 2 \cdot 25 \times 6 + 18 \cdot 75 \times 0 \cdot 5 + 1 \times 6 + 2 \times 3,$$

$$Z_3 \text{ (max.)} = -3 \cdot 32 \text{ tons.}$$

FIG. 164.

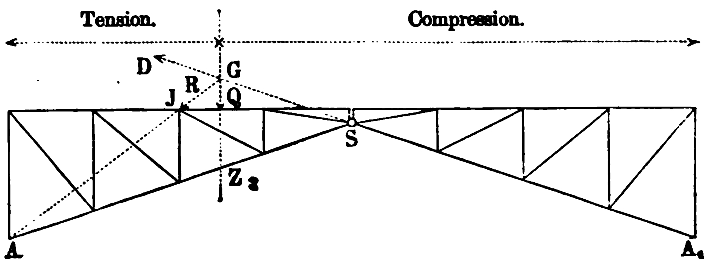
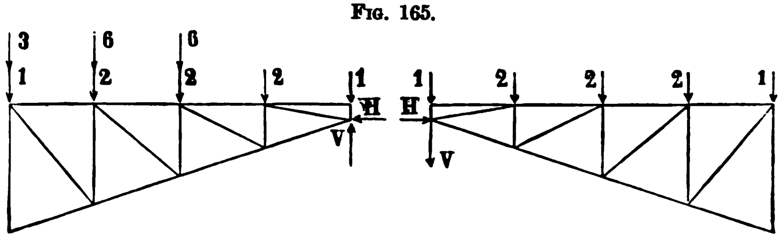


FIG. 165.



To determine Z_3 (min.) the hinge-reaction produced by the loading shown in Fig. 167 must be found from the equations

$$0 = V \times 12 + H \times 4 - 1 \times 12 - 2(9 + 6 + 3) - 3 \times 12 - 6(9 + 6 + 3),$$

$$0 = V \times 12 - H \times 4 + 1 \times 12 + 2(9 + 6 + 3) + 3 \times 12 + 6 \times 9,$$

$$V = 2 \cdot 25, \qquad H = 41 \cdot 25;$$

and the equation of moments for the part of the arch shown in Fig. 168, is therefore

$$0 = Z_3 \times 2 \cdot 37 + 2 \cdot 25 \times 6 + 41 \cdot 25 \times 0 \cdot 5 + 4 \times 6 + 8 \times 3.$$
$$Z_3 \text{ (min.)} = - 34 \cdot 6 \text{ tons.}$$

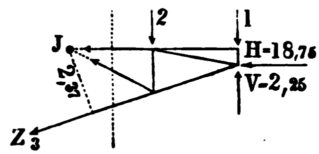
Similarly the stresses in the remaining bars Z can be calculated as follows:

Z_1.

The loading boundary is at the 1st joint, therefore only the minimum stress need be considered. For this

$$V = 0, \qquad H = 48,$$
$$0 = Z_1 \times 4 \cdot 27 + 48 \times 0 \cdot 5 + 4 \times 12$$
$$+ 8 (9 + 6 + 3).$$
$$Z_1 \text{ (min.)} = - 50 \cdot 6 \text{ tons.}$$

FIG. 166.



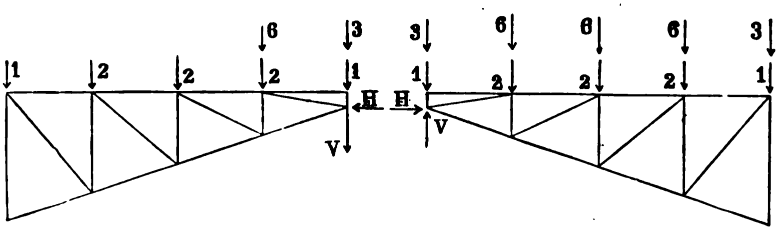
Z_2.

(Loading boundary between the 2nd and 3rd joints.)

For the maximum stress,

$$V = 0 \cdot 75, \qquad H = 14 \cdot 25,$$
$$0 = Z_2 \times 3 \cdot 32 - 0 \cdot 75 \times 9 + 14 \cdot 25 \times 0 \cdot 5 + 1 \times 9 + 2 (6 + 3).$$
$$Z_2 \text{ (max.)} = - 8 \cdot 25 \text{ tons.}$$

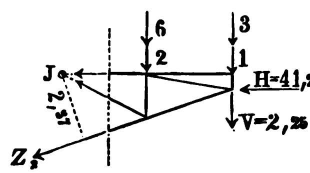
FIG. 167.



For the minimum stress,

$$V = 0 \cdot 75, \qquad H = 45 \cdot 75,$$
$$0 = Z_2 \times 3 \cdot 32 + 0 \cdot 75 \times 9 + 45 \cdot 75 \times 0 \cdot 5$$
$$+ 4 \times 9 + 8 (6 + 3).$$
$$Z_2 \text{ (min.)} = - 41 \cdot 45 \text{ tons.}$$

FIG. 168.



Z_4.

(Loading boundary between the 2nd and 3rd joints.)

For the maximum stress,

$$V = 4 \cdot 5, \qquad H = 25 \cdot 5,$$
$$0 = Z_4 \times 1 \cdot 423 - 4 \cdot 5 \times 3 + 25 \cdot 5 \times 0 \cdot 5 + 1 \times 3.$$
$$Z_4 \text{ (max.)} = - 1 \cdot 58 \text{ tons.}$$

For the minimum stress,

$$V = 4 \cdot 5, \qquad H = 34 \cdot 5,$$

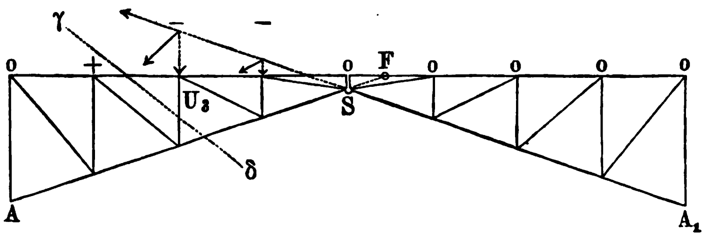
$$0 = Z_4 \times 1 \cdot 423 + 4 \cdot 5 \times 3 + 34 \cdot 5 \times 0 \cdot 5 + 4 \times 3.$$

$$Z_4 \text{ (min.)} = - 30 \cdot 0 \text{ tons.}$$

Calculation of the Stresses U in the Vertical Bars.

The bar U_3 will be chosen to exemplify the method. $\gamma\,\delta$ is the section line, and F is the point about which to take moments. As in the case of Y_3 the loads divide themselves into three groups relatively to the kind of stress produced in

FIG. 169.



U_3, and in Fig. 169 these groups are indicated by the signs $+, -,$ and 0. The stress in U_3 reaches its maximum value when the second joint alone is loaded, and the hinge-reaction can then be obtained from Fig. 162. The values already found for its components are:

$$V = 0 \cdot 75, \qquad H = 14 \cdot 25,$$

and the equation of moments from Fig. 170 is:

$$0 = - U_3 \times 7 \cdot 5 + 0 \cdot 75 \times 1 \cdot 5 + 14 \cdot 25 \times 0 \cdot 5 - 1 \times 1 \cdot 5 - 2\,(4 \cdot 5 + 7 \cdot 5).$$

$$U_3 \text{ (max.)} = - 2 \cdot 3 \text{ tons.}$$

When U_3 is a minimum the third and fourth joints are alone loaded, and the values of the components of the hinge-reaction already found from Fig. 160 are:

$$V = 3 \cdot 75, \qquad H = 23 \cdot 25;$$

and from Fig. 171 the equation of moments is:

$$0 = - U_3 \times 7 \cdot 5 + 3 \cdot 75 \times 1 \cdot 5 + 23 \cdot 25 \times 0 \cdot 5 - 1 \times 1 \cdot 5 - 8\,(4 \cdot 5 + 7 \cdot 5).$$

$$U_3 \text{ (min.)} = - 10 \cdot 7 \text{ tons.}$$

In a similar manner the remaining stresses in the bars U can be found.

U_1.

U_1 (max.) need not be considered, for tension cannot be produced in this bar by any distribution of the loads.

U_1 (min.) obtains when all the joints are loaded, then

$$V = 0, \qquad H = 48,$$

and

$$0 = -U_1 \times 13{\cdot}5 + 48 \times 0{\cdot}5 - 4 \times 1{\cdot}5 - 8\,(4{\cdot}5 + 7{\cdot}5 + 10{\cdot}5) - 4 \times 13{\cdot}5.$$
$$U_1 \text{ (min.)} = -16{\cdot}0 \text{ tons.}$$

U_2.

Here again U_2 (max.) need not be considered, and U_1 (min.) obtains when every joiut is loaded, then

$$V = 0, \qquad H = 48,$$

and

$$0 = -U_2 \times 10{\cdot}5 + 48 \times 0{\cdot}5 - 4 \times 1{\cdot}5 - 8\,(4{\cdot}5 + 7{\cdot}5 + 10{\cdot}5).$$
$$U_2 \text{ (min.)} = -15{\cdot}4 \text{ tons.}$$

Fig. 170.

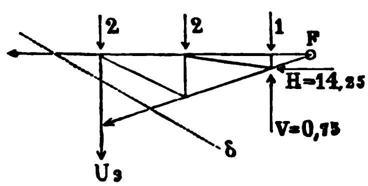
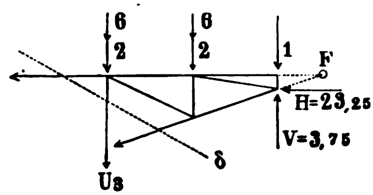


Fig. 171.



U_4.

U_4 (max.) occurs when the 2nd and 3rd joints are loaded, and the 4th unloaded, then

$$V = 2{\cdot}25, \qquad H = 18{\cdot}75,$$

and

$$0 = -U_4 \times 4{\cdot}5 + 2{\cdot}25 \times 1{\cdot}5 + 18{\cdot}75 \times 0{\cdot}5 - 1 \times 1{\cdot}5 - 2 \times 4{\cdot}5.$$
$$U_4 \text{ (max.)} = +\,0{\cdot}5 \text{ tons.}$$

U_1 (min.) occurs when the 4th joint is loaded and the 2nd and 3rd unloaded, then

$$V = 2{\cdot}25 \qquad H = 18{\cdot}75,$$

and

$$0 = -U_4 \times 4{\cdot}5 + 2{\cdot}25 \times 1{\cdot}5 + 18{\cdot}75 \times 0{\cdot}5 - 1 \times 1{\cdot}5 - 8 \times 4{\cdot}5.$$
$$U_4 \text{ (min.)} = -\,5{\cdot}5 \text{ tons.}$$

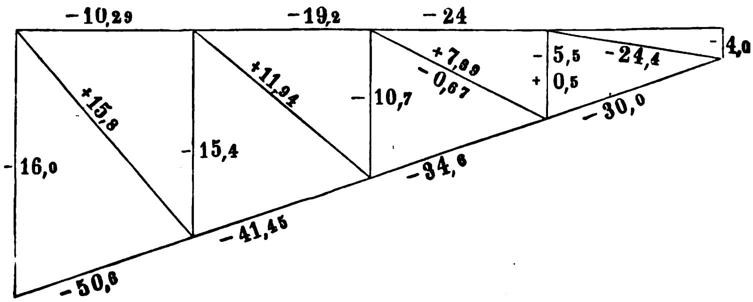
U_5.

The 5th vertical is divided in two at the hinge, and since the head of this vertical is only connected with a horizontal bar, it follows that the only vertical force that can come upon it is the load on the joint, which can never be more than 4 tons for each half. Hence

$$U_5 \text{ (min.)} = -4 \text{ tons.}$$

The results obtained are collected together in Fig. 172.

FIG. 172.



§ 22.—Braced Arch of 40 Metres Span.

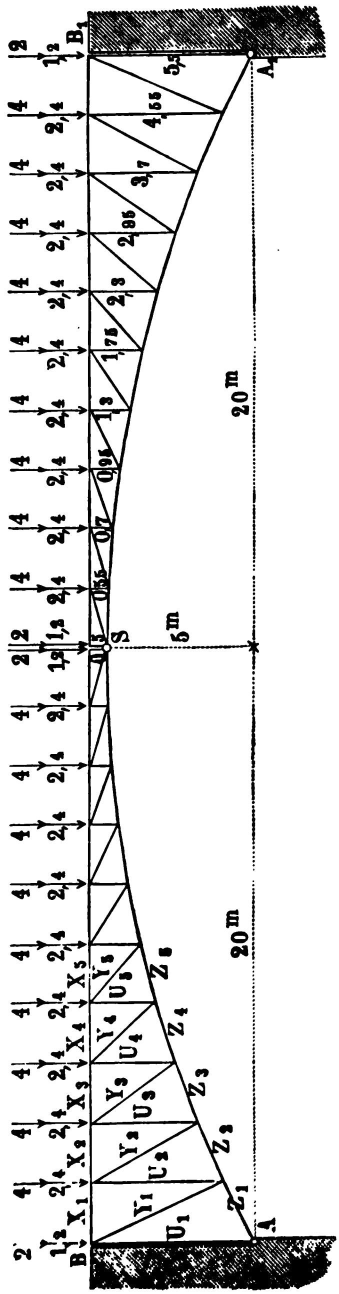
(Bridge over the Theiss at Szegedin.)*

This bridge, supported by two braced arches, is designed for a single line of railway. The permanent load can be taken at 2400 kilos. and the moving load at 4000 kilos. per metre run, and one-half of this is supported by each arch.

The length of a bay being 2 metres, each joint has 2400 kilos. permanent load and 4000 kilos. moving load to bear, or (taking 1000 kilos. = 1 ton) 2·4 tons permanent and 4 tons moving load. The two halves of the arch are connected together at the centre by a hinge, and hinges are also provided at the abutments. The form and dimensions of the structure are given in Fig. 173.

* With the exception of some slight alterations in the dimensions and the addition of a central hinge, the figure represents the bridge over the Theiss. The object of the central hinge will appear in the "Theory of Braced Arches." That the diagonals in the central bays of the existing bridge have been expanded into a plate-web can hardly be considered a difference in the principle of the construction.

FIG. 173.



Calculation of the Stresses X in the Horizontal Bars.

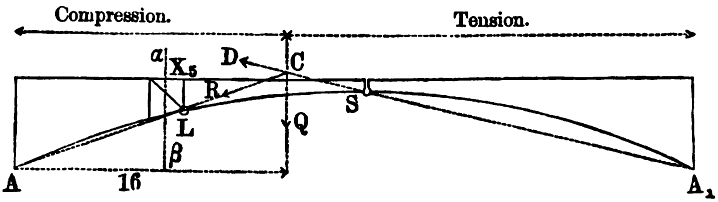
The bar X_5 will be taken to illustrate the calculations. The first step is to determine which loads create tension in X_5 and which compression, and to do this the point must be found where a load can be placed so as to produce no stress in X_5. The vertical through the intersection of A L and A_1 S produced (Fig. 174) gives this required loading boundary, for a load Q placed in this position produces a hinge-reaction D acting in the direction A_1 S (this reaction must pass through A_1 to prevent rotation round that point), and these together give a resultant R which must be directed to A, so that the left half of the arch may not rotate round this point, but by construction the line C A passes through L, and as this is the point about which to take moments to determine X_5, it follows that the load Q can produce no stress in X_5. The reaction R, due to a load to the right of Q passes below L and tends therefore to turn the part L S of the arch from left to right, and X_5 will be positive since it acts in the opposite direction.

On the contrary, every load situated to the left of Q will

produce a negative stress in X_5, for the resultant R in this case passes above L, and therefore tends to produce rotation in the same direction as X_5 does.

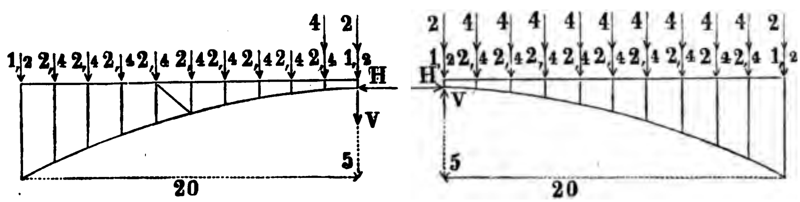
The vertical through C is therefore the required loading boundary, and the maximum stress obtains when all the joints to the right of it are loaded.

FIG. 174.



For X_5 this loading boundary is situated at a distance of 16 metres from the left abutment, coinciding therefore with the vertical bar U_9. To find X_5 (max.) therefore the joints 10, 11, 12—21 must be loaded, and the remainder unloaded

FIG. 175.



(Fig. 175), and the components of the hinge-reaction for this loading are obtained from the two following equations of moments:

$$0 = V \times 20 + H \times 5 - 2{\cdot}4\left(\tfrac{20}{2} + 18 + 16 + \ldots + 4 + 2\right)$$
$$- 4\left(\tfrac{20}{2} + 18 + 16 + \ldots + 4 + 2\right)$$

$$0 = V \times 20 - H \times 5 + 2{\cdot}4\left(\tfrac{20}{2} + 18 + 16 + \ldots + 4 + 2\right)$$
$$+ 4\left(\tfrac{20}{2} + 18\right)$$

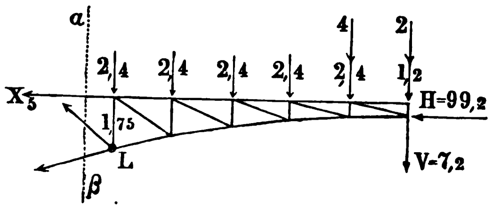
whence

$$V = 7{\cdot}2 \quad \text{and} \quad H = 99{\cdot}2;$$

and the equation of moments for the part of the arch shown in Fig. 176 about L is therefore:

$$0 = -X_5 \times 1\cdot75 - 99\cdot2 \times 1\cdot25 + 7\cdot2 \times 10$$
$$+ 2\cdot4\left(\tfrac{10}{2} + 8 + 6 + 4 + 2\right) + 4\left(\tfrac{10}{2} + 8\right)$$
$$X_5 \text{ (max.)} = +34\cdot29 \text{ tons.}$$

FIG. 176.



To find X_5 (min.) the hinge-reaction must be determined from Fig. 177 thus:

$$0 = -V \times 20 + H \times 5 - 2\cdot4\left(\tfrac{20}{2} + 18 + \ldots + 2\right)$$
$$0 = -V \times 20 - H \times 5 + 2\cdot4\left(\tfrac{20}{2} + 18 + \ldots + 2\right)$$
$$+ 4\,(14 + 12 + \ldots + 2)$$
$$V = 5\cdot6 \qquad H = 70\cdot4;$$

FIG. 177.

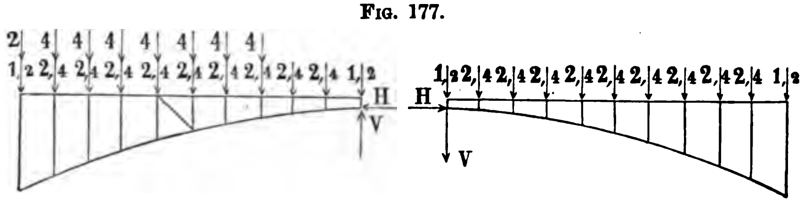
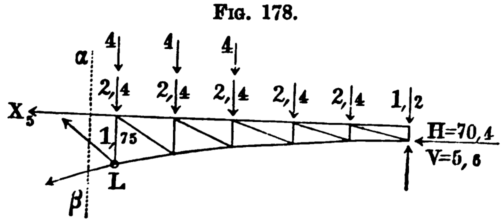


FIG. 178.



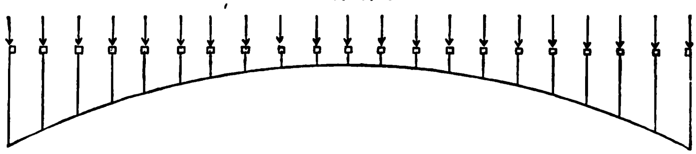
and the equation of moments is (Fig. 178):

$$0 = -X_5 \times 1\cdot75 - 5\cdot6 \times 10 - 70\cdot4 \times 1\cdot25 + 2\cdot4\left(\tfrac{10}{2} + 8 + \ldots + 2\right)$$
$$+ 4\,(4 + 2)$$
$$X_5 \text{ (min.)} = -34\cdot29 \text{ tons.}$$

From this it appears that the numerical values of X_5 (max.) and X_5 (min.) are identical, whence it follows that $X_5 = 0$, when the loads producing the maximum stress are on the bridge together with those producing the minimum stress, that is when the bridge is fully loaded (for the load on the 9th joint has no effect). This property is easily explained by the "Theory of parabolic girders," given in § 8, for in the present example the arch has the form of a parabola, and it has been shown that this is the curve of equilibrium (or linear arch) for a load uniformly distributed over the span. Directly therefore the bridge is fully loaded, neither the horizontal bars nor the diagonals are necessary to maintain equilibrium, the verticals, however, are required to transmit the loads to the linear arch (Fig. 179).

Now the permanent load is uniformly distributed over the span, and produces therefore no stress in the horizontal bars or the diagonals. Thus in calculating the stresses in them, the

FIG. 179.



permanent load can be left out of consideration, and further it is only necessary to obtain either the maximum or minimum stress when the other can be found by changing the sign.

The calculation for X_5 could therefore have been given in the following form.

$$0 = -V \times 20 + H \times 5$$
$$0 = -V \times 20 - H \times 5 + 4\,(14 + 12 + \ldots + 2)$$
$$V = 5{\cdot}6 \qquad H = 22{\cdot}4$$
$$0 = -X_5 \times 1{\cdot}75 - 5{\cdot}6 \times 10 - 22{\cdot}4 \times 1{\cdot}25 + 4\,(4 + 2)$$
$$X_5 = \pm\, 34{\cdot}29 \text{ tons.}$$

And the following calculations are made in a similar manner:

X_1.

(Loading boundary in 7th bay.)

$$0 = -V \times 20 + H \times 5$$
$$0 = -V \times 20 - H \times 5 + 4\,(12 + 10 + \ldots + 2)$$
$$V = 4{\cdot}2 \qquad H = 16{\cdot}8$$
$$0 = -X_1 \times 4{\cdot}55 - 4{\cdot}2 \times 18 - 16{\cdot}8 \times 4{\cdot}05 + 4\,(10 + 8 + \ldots + 2)$$
$$X = \pm\, 5{\cdot}20 \text{ tons.}$$

X_2.

(Loading boundary in 8th bay.)

$$0 = - V \times 20 + H \times 5$$
$$0 = - V \times 20 - H \times 5 + 4(14 + 12 + \ldots + 2)$$
$$V = 5 \cdot 6 \qquad H = 22 \cdot 4$$
$$0 = - X_2 \times 3 \cdot 7 - 5 \cdot 6 \times 16 - 22 \cdot 4 \times 3 \cdot 2 + 4(10 + \ldots + 2)$$
$$X_2 = \pm 11 \cdot 16 \text{ tons.}$$

X_3.

(Loading boundary in 8th bay.)

$$V = 5 \cdot 6 \qquad H = 22 \cdot 4$$
$$0 = - X_3 \times 2 \cdot 95 - 5 \cdot 6 \times 14 - 22 \cdot 4 \times 2 \cdot 45 + 4(8 + 6 + \ldots + 2)$$
$$X_3 = \pm 18 \cdot 06 \text{ tons.}$$

X_4.

(Loading boundary in 8th bay.)

$$V = 5 \cdot 6 \qquad H = 22 \cdot 4$$
$$0 = - X_4 \times 2 \cdot 3 - 5 \cdot 6 \times 12 - 22 \cdot 4 \times 1 \cdot 8 + 4(6 + 4 + 2)$$
$$X_4 = \pm 25 \cdot 88 \text{ tons.}$$

X_6.

(Loading boundary in 9th bay.)

$$0 = - V \times 20 + H \times 5$$
$$0 = - V \times 20 - H \times 5 + 4(16 + 14 + \ldots + 2)$$
$$V = 7 \cdot 2 \qquad H = 28 \cdot 8$$
$$0 = - X_6 \times 1 \cdot 3 - 7 \cdot 2 \times 8 - 28 \cdot 8 \times 0 \cdot 8 + 4(4 + 2)$$
$$X_6 = \pm 43 \cdot 57 \text{ tons.}$$

X_7.

(Loading boundary in 9th bay.)

$$V = 7 \cdot 2 \qquad H = 28 \cdot 8$$
$$0 = - X_7 \times 0 \cdot 95 - 7 \cdot 2 \times 6 - 28 \cdot 8 \times 0 \cdot 45 + 4 \times 2$$
$$X_7 = \pm 50 \cdot 70 \text{ tons.}$$

X_8.

(Loading boundary in 10th bay.)

$$0 = - V \times 20 + H \times 5$$
$$0 = - V \times 20 - H \times 5 + 4(18 + 16 + \ldots + 2)$$
$$V = 9 \qquad H = 36$$
$$0 = - X_8 \times 0 \cdot 7 - 9 \times 4 - 36 \times 0 \cdot 2 + 4 \times 2$$
$$X_8 = \pm 50 \cdot 29 \text{ tons.}$$

X_9.

(Loading boundary in 10th bay.)

$$V = 9 \qquad H = 36$$
$$0 = - X_9 \times 0 \cdot 55 - 9 \times 2 - 36 \times 0 \cdot 05$$
$$X_9 = \pm 36 \cdot 0 \text{ tons.}$$

X_{10}.

There never can be any stress in this bar, for no horizontal force can act on its right extremity. Hence

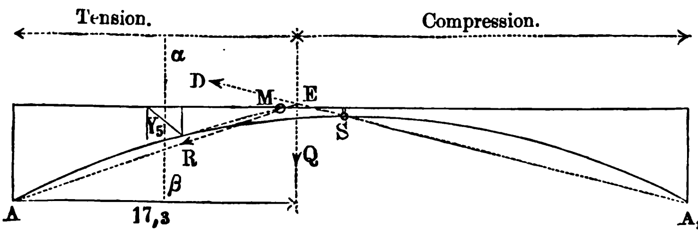
$$X_{10} = 0.$$

Calculation of the Stresses Y in the Diagonals.

The diagonal marked Y_5 will serve to illustrate the calculations.

The point about which moments will be taken is M (Fig. 180), and the vertical through the intersection of A M and A_1 S produced will give the loading boundary. For a load Q placed in this position gives with the hinge-reaction D a resultant R whose direction is E M A. If the load lies to

FIG. 180.



the right of E the resultant R, or if it is placed on the right half of the arch the hinge-reaction passes below M, and consequently tends to turn the part S α β from left to right, but Y_5 has the same tendency, and must therefore be negative.

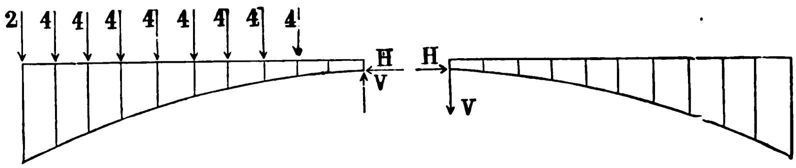
If on the contrary the load is to the left of E, the resultant R, or if the load is to the left of α β, the hinge-reaction D passes above M, and in either case Y_5 is evidently positive.

Thus if the part of the bridge lying to the right of the vertical through E be loaded, Y_5 will be a minimum; and if the part to the left be loaded, Y_5 will be a maximum.

It has already been remarked when dealing with the stresses in the horizontal bars, that the permanent load produces no stress in the diagonals; it seems therefore unnecessary to carry the proof any further. Hence in the following calculations

the permanent load will be left out of consideration (Fig. 181). Also since the numerical values of the maxima and minima stresses are equal, only the maximum stress will be calculated in each case, the minimum stress being obtained by changing the sign.

FIG. 181.



Thus to determine Y_5 (max.) the equations to find the hinge-reaction are (Fig. 182)

$$0 = -V \times 20 + H \times 5$$
$$0 = -V \times 20 - H \times 5 + 4\,(16 + 14 + \ldots + 2)$$
$$V = 7\cdot 2 \qquad H = 28\cdot 8$$

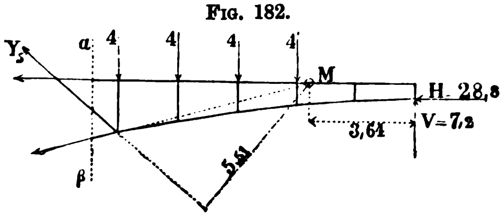
and from the same figure the equation of moments is

$$0 = Y_5 \times 5\cdot 51 - 7\cdot 2 \times 3\cdot 64 + 28\cdot 8 \times 0\cdot 5$$
$$- 4\,(0\cdot 36 + 2\cdot 36 + 4\cdot 36 + 6\cdot 36)$$
$$Y_5\ (\text{max.}) = +\ 11\cdot 9 \text{ tons}$$

and

$$Y_5\ (\text{min.}) = -\ 11\cdot 9 \text{ tons.}$$

FIG. 182.



In a similar manner the stresses in the remaining diagonals can be found as follows:

Y_1.

(Loading boundary in 7th bay.)

$$V = 4\cdot 2 \qquad H = 16\cdot 8 \quad (\text{see calc. for } X_1)$$
$$0 = Y_1 \times 10\cdot 6 - 4\cdot 2 \times 8\cdot 42 + 16\cdot 8 \times 0\cdot 5 + 4 \times 0\cdot 42$$
$$- 4\,(1\cdot 58 + 3\cdot 58 + 5\cdot 58 + 7\cdot 58 + 9\cdot 58)$$

·2 tons.

Y_2.

(Loading boundary in 8th bay.)

$$V = 5 \cdot 6 \qquad H = 22 \cdot 4 \quad \text{(see calc. for } X_2\text{)}$$

$$0 = Y_2 \times 9 \cdot 42 - 5 \cdot 6 \times 7 \cdot 294 + 22 \cdot 4 \times 0 \cdot 5 + 4 \times 1 \cdot 294$$
$$- 4\,(0 \cdot 706 + 2 \cdot 706 + 4 \cdot 706 + 6 \cdot 706 + 8 \cdot 706)$$
$$Y_2 = \pm\, 12 \cdot 59 \text{ tons.}$$

Y_3.

(Loading boundary in 8th bay.)

$$V = 5 \cdot 6 \qquad H = 22 \cdot 4$$

$$0 = Y_3 \times 8 \cdot 16 - 5 \cdot 6 \times 6 \cdot 13 + 22 \cdot 4 \times 0 \cdot 5 + 4 \times 0 \cdot 13$$
$$- 4\,(1 \cdot 87 + 3 \cdot 87 + 5 \cdot 87 + 7 \cdot 87)$$
$$Y_3 = \pm\, 12 \cdot 3 \text{ tons.}$$

Y_4.

(Loading boundary in 9th bay.)

$$V = 7 \cdot 2 \qquad H = 28 \cdot 8 \quad \text{(see calc. for } X_6\text{)}$$

$$0 = Y_4 \times 6 \cdot 834 - 7 \cdot 2 \times 4 \cdot 923 + 28 \cdot 8 \times 0 \cdot 5 + 4 \times 0 \cdot 923$$
$$- 4\,(1 \cdot 077 + 3 \cdot 077 + 5 \cdot 077 + 7 \cdot 077)$$
$$Y_4 = \pm\, 12 \cdot 07 \text{ tons.}$$

Y_6.

(Loading boundary in 9th bay.)

$$V = 7 \cdot 2 \qquad H = 28 \cdot 8$$

$$0 = Y_6 \times 4 \cdot 24 - 7 \cdot 2 \times 2 \cdot 223 + 28 \cdot 8 \times 0 \cdot 5$$
$$- 4\,(1 \cdot 777 + 3 \cdot 777 + 5 \cdot 777)$$
$$Y_6 = \pm\, 11 \cdot 07 \text{ tons.}$$

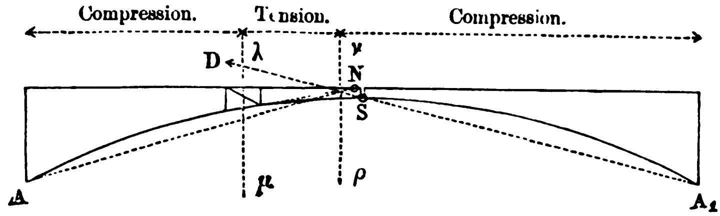
In the case of Y_7 it is found that the point about which to take moments is situated in the central bay, and this, as will appear, makes the arrangement of the loading giving the greatest stresses differ from that of the previous cases (Fig. 183). There are, in fact, three groups of loads, two of these produce compression and the third tension. For as will be seen from the figure the line A_1 S in this case passes below the point N, and consequently the hinge-reaction produced by a load on the left half of the arch also passes below N. Now a load situated to the left of the section line $\lambda\ \mu$ acts on the part of the arch S $\lambda\ \mu$ by its hinge-reaction D, and tends therefore to turn this part round N from left to right, thus making Y_7 negative.

Thus the section line $\lambda\ \mu$ is a second loading boundary;

for, as in the former cases, a load placed between $\lambda\ \mu$ and $\gamma\ \rho$ produces tension in Y_7.

Hence, to find the greatest stress in Y_7, either the two compression groups can be considered loaded or else the tension group.

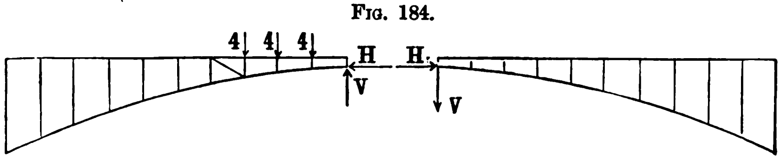
Fig. 183.



In the latter case the equations to find the hinge-reaction are (Fig. 184),

$$0 = -V \times 20 + H \times 5$$
$$0 = -V \times 20 - H \times 5 + 4\,(18 + 16 + 14)$$
$$V = 4 \cdot 8 \qquad H = 19 \cdot 2$$

Fig. 184.

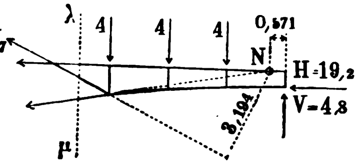


and the equation of moments from Fig. 185 is,

$$0 = Y_7 \times 3 \cdot 194 - 4 \cdot 8 \times 0 \cdot 571 + 19 \cdot 2 \times 0 \cdot 5$$
$$- 4\,(1 \cdot 429 + 3 \cdot 429 + 5 \cdot 429)$$
$$Y_7 = \pm 10 \cdot 73 \text{ tons.}$$

For Y_8 the loads also form three groups. The turning point is in this case situated in the right centre bay and the hinge-reaction D passes below F. The calculations are exactly similar to those for Y_7, and

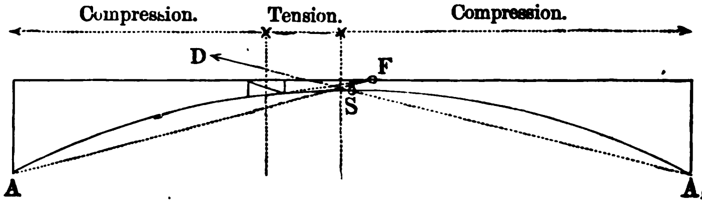
Fig. 185.



$$0 = -V \times 20 + H \times 5$$
$$0 = -V \times 20 - H \times 5 + 4\,(18 + 16)$$
$$V = 3 \cdot 4 \qquad H = 13 \cdot 6$$
$$0 = Y_8 \times 2 \cdot 51 + 3 \cdot 4 \times 1 \cdot 6 + 13 \cdot 6 \times 0 \cdot 5 - 4\,(5 \cdot 6 + 3 \cdot 6)$$
$$Y_8 = \pm 9 \cdot 8 \text{ tons.}$$

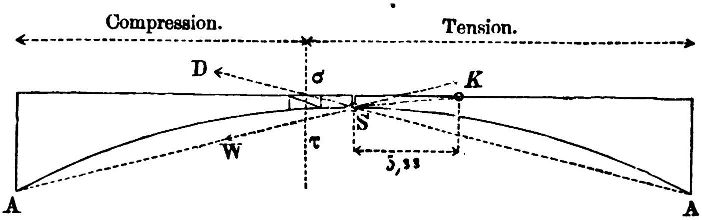
For Y_9, however, the position of the turning point is such that only two groups are formed (Fig. 187). Here the section line $\sigma\,\tau$ is itself the loading boundary, for every load to the left of $\sigma\,\tau$ acts on the part $S\,\sigma\,\tau$ through its hinge-reaction D, which

FIG. 186.



evidently makes Y_7 negative. But every load to the right of $\sigma\,\tau$ on the part $S\,\sigma\,\tau$, produces with its hinge-reaction a resultant which tends to induce rotation from right to left, and all loads on the right half of the arch acting by means of

FIG. 187.



their hinge-reaction W have the same effect. Consequently all loads to the right of $\sigma\,\tau$ make Y_9 positive.

The loading boundary is therefore situated in the 9th bay, and (see calculation for X_6)

$$V = 7 \cdot 2 \qquad H = 28 \cdot 8$$

Whence the equation of moments is

$$0 = Y_9 \times 2 \cdot 47 + 7 \cdot 2 \times 5 \cdot 33 + 28 \cdot 8 \times 0 \cdot 5$$
$$Y_9 = \pm\, 21 \cdot 4 \text{ tons.}$$

Similarly it is found that the section line is the loading boundary for Y_{10}, therefore,

$$0 = -V \times 20 + H \times 5$$
$$0 = -V \times 20 - H \times 5 + 4\,(18 + 16 + \ldots + 2)$$
$$V = 9 \qquad H = 36$$
$$0 = Y_{10} \times 5 \cdot 324 + 9 \times 20 + 36 \times 0 \cdot 5$$
$$Y_{10} = \pm\, 37 \cdot 29 \text{ tons.}$$

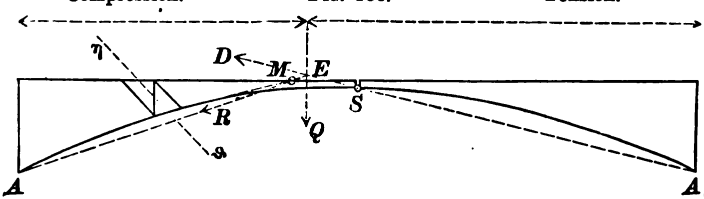
Calculation of the Stresses U in the Verticals.

The effect of the permanent load on the vertical bars can be deduced from Fig. 179. If half the permanent load is applied to the top of the verticals and the other half to the foot, the compression produced in each vertical will be 1·2 tons (with the exception, however, of the first and last verticals which have only half the amount to sustain). To these stresses must now be added those produced by the moving load.

The maximum and minimum stresses produced by the moving load must therefore be found.

The vertical U_5 will be taken to illustrate the method. The loading boundary can be found by the construction employed in Fig. 180, for in both cases M is the turning point.

Compression. FIG. 188. Tension.



A load Q placed on the vertical through E, gives with its hinge-reaction D a resultant R which passes through M, and hence Q can produce no stress in U_5. The vertical through E is therefore the loading boundary, and all loads to the right produce tension, and all loads to the left compression. When U_5 (min.) obtains the bridge will be loaded with the compression group, and then (see calculations for Y_5)

$$V = 7{\cdot}2 \qquad H = 28{\cdot}8$$

Whence the equation of moments for the part of the arch shown in Fig. 189 is (denoting by u_5 the stress due to the moving load alone),

$$0 = - u_5 \times 8{\cdot}36 - 7{\cdot}2 \times 3{\cdot}64 + 28{\cdot}8 \times 0{\cdot}5$$
$$- 4\,(0{\cdot}36 + 2{\cdot}36 + 4{\cdot}36 + 6{\cdot}36 + 8{\cdot}36)$$
$$u_5 \text{ (min.)} = - 11{\cdot}84 \text{ tons.}$$

u_5 (max.) can be deduced from this without further calculation in the following manner:

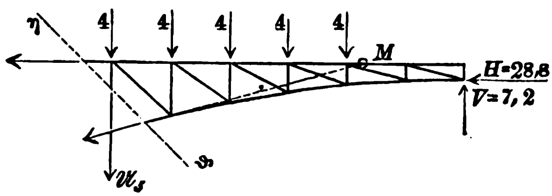
If the moving load covers the whole bridge it is evident that the stress in each vertical is -4 tons, and hence u_5 (max.) together with u_5 (min.) must be equal to -4 tons, or

$$u_5 \text{ (max.)} = -4 - (-11\cdot84) = +7\cdot84 \text{ tons.}$$

To obtain U_5 (max.) and U_5 (min.) $-1\cdot2$ tons must be added to the values just found, thus:

$$U_5 \text{ (min.)} = -11\cdot84 - 1\cdot2 = -13\cdot04 \text{ tons}$$
$$U_5 \text{ (max.)} = +\ 7\cdot84 - 1\cdot2 = +\ 6\cdot64 \text{ tons.*}$$

FIG. 189.



The stresses in the remaining verticals can be found as follows

U_1.†

$$0 = -u_1 \times 11\cdot58 - 4\cdot2 \times 8\cdot42 + 16\cdot8 \times 0\cdot5 + 4 \times 0\cdot42$$
$$-4\,(1\cdot58 + 3\cdot58 + 5\cdot58 + 7\cdot58 + 9\cdot58 + 11\cdot58)$$
$$u_1 \text{ (min.)} = -15\cdot82 \qquad u_1 \text{ (max.)} = +11\cdot82$$
$$U_1 \text{ (min.)} = -17\cdot02 \text{ tons}$$
$$U_1 \text{ (max.)} = +10\cdot62 \text{ tons.}$$

U_2.

$$0 = -u_2 \times 10\cdot706 - 5\cdot6 \times 7\cdot294 + 22\cdot4 \times 0\cdot5 + 4 \times 1\cdot294$$
$$-4\,(0\cdot706 + 2\cdot706 + 4\cdot706 + 6\cdot706 + 8\cdot706 + 10\cdot706)$$
$$u_2 \text{ (min.)} = -15\cdot08 \qquad u_2 \text{ (max.)} = +11\cdot08$$
$$U_2 \text{ (min.)} = -16\cdot28 \text{ tons}$$
$$U_2 \text{ (max.)} = +\ 9\cdot88 \text{ tons.}$$

* These stresses could have been obtained quicker thus: To find U_5 (min.) add to the vertical component of Y_5 (min.) $-(4 + 1\cdot2)$ tons, and to find U_5 (max.), add to the vertical component of Y_5 (max.) $-1\cdot2$ tons. As, however, this method cannot always be adopted, the longer one has been preferred.

† Strictly speaking, the load on the 1st vertical is only half the load on the others, the remaining half being taken by the abutment. It has, however, been *considered fully loaded*, as this would probably be the course pursued in practice.

U_3.

$$0 = -u_3 \times 9\cdot87 - 5\cdot6 \times 6\cdot13 + 22\cdot4 \times 0\cdot5 + 4 \times 0\cdot13 - 4\,(1\cdot87 + 3\cdot87 + 5\cdot87 + 7\cdot87 + 9\cdot87)$$

$$u_3\,(\text{min.}) = -14\cdot2 \qquad u_3\,(\text{max.}) = +10\cdot2$$

$$U_3\,(\text{min.}) = -15\cdot4 \text{ tons}$$
$$U_3\,(\text{max.}) = +\ 9\cdot0 \text{ tons.}$$

U_4.

$$0 = -u_4 \times 9\cdot077 - 7\cdot2 \times 4\cdot923 + 28\cdot8 \times 0\cdot5 + 4 \times 0\cdot923 - 4\,(1\cdot077 + 3\cdot077 + 5\cdot077 + 7\cdot077 + 9\cdot077)$$

$$u_4\,(\text{min.}) = -13\cdot1 \qquad u_4\,(\text{max.}) = +9\cdot1$$

$$U_4\,(\text{min.}) = -14\cdot3 \text{ tons}$$
$$U_4\,(\text{max.}) = +\ 7\cdot9 \text{ tons.}$$

U_6.

$$0 = -u_6 \times 7\cdot777 - 7\cdot2 \times 2\cdot223 + 28\cdot8 \times 0\cdot5 - 4\,(1\cdot777 + 3\cdot777 + 5\cdot777 + 7\cdot777)$$

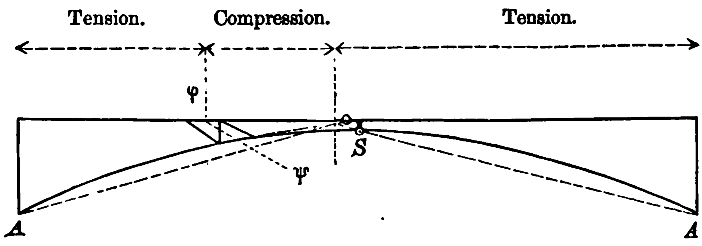
$$u_6\,(\text{min.}) = -10\cdot03 \qquad u_6\,(\text{max.}) = +6\cdot03$$

$$U_6\,(\text{min.}) = -11\cdot23 \text{ tons}$$
$$U_6\,(\text{max.}) = +\ 4\cdot83 \text{ tons.}$$

U_7.

In determining the stress in Y_7 it was found that the loads divided themselves into three groups. This is also true in the case of U_7 with this difference, that the second loading boundary is placed one bay more to the left on account of the oblique direction of the section line $\phi\ \psi$ (Fig. 190).

FIG. 190.



When the loads producing compression are on the arch the equations to obtain the components of the hinge-reaction are,

$$0 = -V \times 20 + H \times 5$$
$$0 = -V \times 20 - H \times 5 + 4\,(18 + 16 + 14 + 12)$$
$$V = 6 \qquad H = 24$$

and the equation of moments for the part $S\phi\psi$ is,

$$0 = -u_7 \times 7\cdot429 - 6 \times 0\cdot571 + 24 \times 0\cdot5$$
$$- 4\,(1\cdot429 + 3\cdot429 + 5\cdot429 + 7\cdot429)$$
$$u_7 \text{ min.}) = -8\cdot38 \qquad u_7 \text{ (max.)} = +4\cdot38$$
$$U_7 \text{ (min.)} = -9\cdot58 \text{ tons}$$
$$U_7 \text{ (max.)} = +3\cdot18 \text{ tons.}$$

U_8.

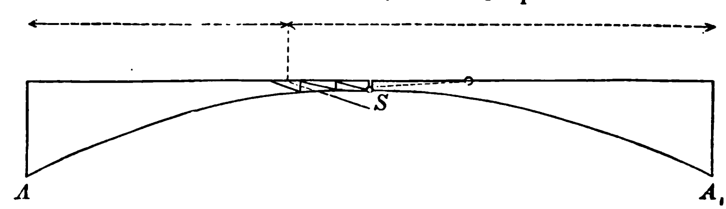
In this case also there are two loading boundaries, and

$$0 = -V \times 20 + H \times 5$$
$$0 = -V \times 20 - H \times 5 + 4\,(18 + 16 + 14)$$
$$V = 4\cdot8 \qquad H = 19\cdot2$$
$$0 = -u_8 \times 7\cdot6 + 4\cdot8 \times 1\cdot6 + 19\cdot2 \times 0\cdot5$$
$$- 4\,(3\cdot6 + 5\cdot6 + 7\cdot6)$$
$$u_8 \text{ (min.)} = -6\cdot57 \qquad u_8 \text{ (max.)} = +2\cdot57$$
$$U_8 \text{ (min.)} = -7\cdot77 \text{ tons}$$
$$U_8 \text{ (max.)} = +1\cdot37 \text{ tons.}$$

U_9.

As in the case of Y_9 the loads divide themselves again into two groups; the loading boundary is however one bay more to the left (Fig. 191).

Tension. FIG. 191. Compression.



To obtain the hinge-reaction,

$$0 = -V \times 20 + H \times 5$$
$$0 = -V \times 20 - H \times 5 + 4\,(14 + 12 + \ldots + 2)$$
$$V = 5\cdot6 \qquad H = 22\cdot4$$

and

$$0 = -u_9 \times 9\cdot33 + 5\cdot6 \times 5\cdot33 + 22\cdot4 \times 0\cdot5$$
$$u_9 \text{ (min.)} = -8\cdot4 \qquad u_9 \text{ (max.)} = +4\cdot4$$
$$U_9 \text{ (min.)} = -9\cdot6 \text{ tons}$$
$$U_9 \text{ (max.)} = +3\cdot2 \text{ tons.}$$

U_{10}.

The loading boundary is in the 9th bay, and

$$V = 7 \cdot 2 \qquad H = 28 \cdot 8$$
$$0 = - u_{10} \times 22 + 7 \cdot 2 \times 20 + 28 \cdot 8 \times 0 \cdot 5$$
$$u_{10}\,(\text{min.}) = - 11 \cdot 2 \qquad u_{10}\,(\text{max.}) = + 7 \cdot 2$$
$$U_{10}\,(\text{min.}) = - 12 \cdot 4 \text{ tons}$$
$$U_{10}\,(\text{max.}) = + \ 6 \cdot 0 \text{ tons.}$$

U_{11}.

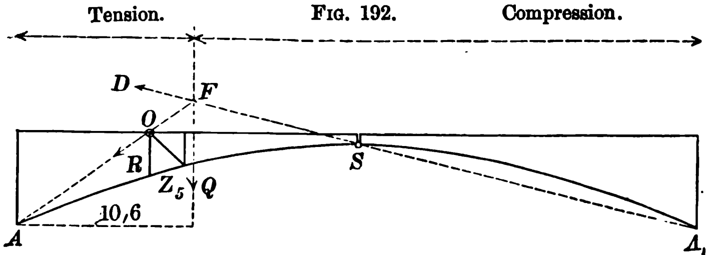
The vertical in the centre is divided in two by the hinge, and as at the top it is only connected to a horizontal bar, the only stress that can exist in it is the compression produced by a load placed on the top. The greatest load for each half is 0·6 ton permanent, and 2 tons moving load. Hence

$$U_{11}\,(\text{min.}) = - 2 \cdot 6 \text{ tons.}$$

Calculation of the Stresses Z in the Bow.

To find the stress in the bar Z_5 a section line is drawn through the 5th bay, and the equation of moments formed for the part of the arch lying between the section line and the hinge with reference to the point of intersection O of the diagonal and horizontal bars (Fig. 192).

FIG. 192.



The vertical through the point of intersection F of A O and A_1 S is the loading boundary, for the resultant R of a load Q in this position and its hinge-reaction D passes through O. This loading boundary lies in the 6th bay.

In this case the permanent load will have to be taken into

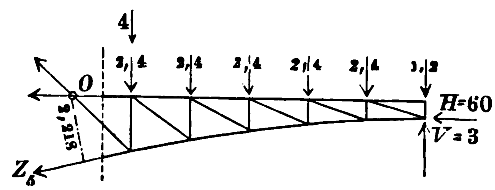
consideration; every joint will therefore have a permanent load of 2·4 tons, and those joints which have been called the loaded joints will carry 4 tons of moving load besides. The distribution of the moving load producing the greatest tension is shown in Fig. 192, and the equations to find the hinge-reaction are,

$$0 = -V \times 20 + H \times 5 - 2{\cdot}4\left(\tfrac{20}{2} + 18 + 16 + \ldots + 2\right)$$
$$0 = -V \times 20 - H \times 5 + 2{\cdot}4\left(\tfrac{20}{2} + 18 + 16 + \ldots + 2\right)$$
$$+ 4\,(10 + 8 + \ldots + 2)$$
$$V = 3 \qquad H = 60.$$

Consequently, from Fig. 193,

$$0 = Z_5 \times 2{\cdot}218 - 3 \times 12 + 60 \times 0{\cdot}5 + 2{\cdot}4\left(\tfrac{12}{2} + 10 + \ldots + 2\right) + 4 \times 2$$
$$Z_5\ (\text{max.}) = -\ 39{\cdot}86 \text{ tons.}$$

Fig. 193.



Again, to find Z_5 (min.) (Fig. 192),

$$0 = V \times 20 + H \times 5 - 6{\cdot}4\left(\tfrac{20}{2} + 18 + \ldots + 2\right)$$
$$0 = V \times 20 - H \times 5 + 2{\cdot}4\left(\tfrac{20}{2} + 18 + \ldots + 2\right) + 4\left(\tfrac{20}{2} + 18 + \ldots + 2\right)$$
$$V = 3 \qquad H = 116$$

and from Fig. 194

$$0 = Z_5 \times 2{\cdot}218 + 3 \times 12 + 116 \times 0{\cdot}5 + 2{\cdot}4\left(\tfrac{12}{2} + 10 + \ldots + 2\right)$$
$$+ 4\left(\tfrac{12}{2} + 10 + \ldots + 4\right)$$
$$Z_5\ (\text{min.}) = -\ 142{\cdot}70 \text{ tons.}$$

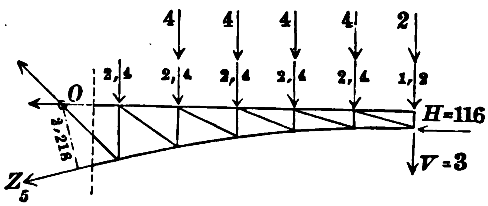
For the sake of comparison the stress in Z_5, when the moving load covers the whole bridge, will also be calculated, thus,

$$0 = V \times 20 + H \times 5 - 6{\cdot}4\left(\tfrac{20}{2} + 18 + \ldots + 2\right)$$
$$0 = V \times 20 - H \times 5 + 6{\cdot}4\left(\tfrac{20}{2} + 18 + \ldots + 2\right)$$
$$V = 0 \qquad H = 128$$
$$0 = Z_5 \times 2{\cdot}218 + 128 \times 0{\cdot}5 + 6{\cdot}4\left(\tfrac{12}{2} + 10 + \ldots + 2\right)$$
$$Z_5 = -\ 132{\cdot}7 \text{ tons.}$$

From the above calculations it appears that the compression in the bars forming the bow can be considerably greater with an uneven than with a uniformly distributed load. In this a parabolic arch-bridge differs from a parabolic girder-bridge, for it was shown that in the latter the greatest compression in the bow occurred when the bridge was fully loaded.

It also appears that it is not absolutely necessary to calculate the maximum stress in the bow. For if the maximum and minimum stresses produced by the moving load be added together the result is the stress due to the moving load when it covers the bridge. And it is evident that this stress is always negative (for altering 6·4 to 4 in the last equation will not change the sign of Z_5). Consequently the absolute value of the minimum stress produced by the moving load must be greater than that of the maximum stress, and the compression

FIG. 194.



produced by the dead load still further increases the balance in favour of the minimum stress. And since a greater section of material is generally required to resist compression than the same amount of tension, the maxima stresses might be neglected.

They will, however, be calculated, and for the following reason. If the arch be imagined turned upside down it becomes a suspension bridge, and the same calculations with reversed signs would apply, the minima stresses becoming the maxima stresses and *vice versâ*. Now it is just possible, if the dead load were small in comparison to the moving load, that the minima stresses in the suspension bridge might become negative, and would then probably determine the section (the bars being long columns). For such cases therefore it is necessary to know what are the maxima stresses in Z.

The stresses in the remaining bars Z can be calculated as follows:

Z_1.

(The loading boundary coincides with the 1st vertical.)

When the bridge is free of the moving load,

$$0 = V \times 20 + H \times 5 - 2\cdot 4\left(\tfrac{20}{2} + 18 + \ldots + 2\right)$$
$$0 = V \times 20 - H \times 5 + 2\cdot 4\left(\tfrac{20}{2} + 18 + \ldots + 2\right)$$
$$V = 0 \qquad H = 48$$
$$0 = Z_1 \times 4\cdot 968 + 48 \times 0\cdot 5 + 2\cdot 4\left(\tfrac{20}{2} + 18 + \ldots + 2\right)$$
$$Z_1 \text{ (max.)} = -53\cdot 14 \text{ tons.}$$

When the moving load covers the bridge,

$$V = 0 \qquad H = 128$$
$$0 = Z_1 \times 4\cdot 968 + 128 \times 0\cdot 5 + 6\cdot 4\left(\tfrac{20}{2} + 18 + \ldots + 2\right)$$
$$Z_1 \text{ (min.)} = -141\cdot 71 \text{ tons.}$$

Z_2.

(Loading boundary in the 2nd bay.)

$$V = 0\cdot 2 \qquad H = 48\cdot 8$$
$$0 = Z_2 \times 4\cdot 186 - 0\cdot 2 \times 18 + 48\cdot 8 \times 0\cdot 5 + 2\cdot 4\left(\tfrac{18}{2} + 16 + \ldots + 2\right)$$
$$Z_2 \text{ (max.)} = -51\cdot 41 \text{ tons.}$$
$$V = 0\cdot 2 \qquad H = 127\cdot 2$$
$$0 = Z_2 \times 4\cdot 186 + 0\cdot 2 \times 18 + 127\cdot 2 \times 0\cdot 5 + 6\cdot 4\left(\tfrac{18}{2} + 16 + \ldots + 2\right)$$
$$Z_2 \text{ (min.)} = -139\cdot 89 \text{ tons.}$$

Z_3.

(Loading boundary in the 4th bay.)

$$V = 1\cdot 2 \qquad H = 52\cdot 8$$
$$0 = Z_3 \times 3\cdot 464 - 1\cdot 2 \times 16 + 52\cdot 8 \times 0\cdot 5 + 2\cdot 4\left(\tfrac{16}{2} + 14 + \ldots + 2\right) + 4 \times 2$$
$$Z_3 \text{ (max.)} = -48\cdot 73 \text{ tons.}$$
$$V = 1\cdot 2 \qquad H = 123\cdot 2$$
$$0 = Z_3 \times 3\cdot 464 + 1\cdot 2 \times 16 + 123\cdot 2 \times 0\cdot 5 + 2\cdot 4\left(\tfrac{16}{2} + 14 + \ldots + 2\right) + 4\left(\tfrac{16}{2} + 14 + \ldots + 4\right)$$
$$Z_3 \text{ (min.)} = -139\cdot 3 \text{ tons.}$$

Z_4.

(Loading boundary in the 5th bay.)

$$V = 2 \qquad H = 56$$

$$0 = Z_4 \times 2{\cdot}805 - 2 \times 14 + 56 \times 0{\cdot}5$$
$$+ 2{\cdot}4\,(\tfrac{14}{2} + 12 + \ldots + 2) + 4 \times 2$$

Z_4 (max.) $= -\ 44{\cdot}77$ tons.

$$V = 2 \qquad H = 120$$

$$0 = Z_4 \times 2{\cdot}805 + 2 \times 14 + 120 \times 0{\cdot}5$$
$$+\ 2{\cdot}4\,(\tfrac{14}{2} + 12 + \ldots + 2)$$
$$+\ 4\,(\tfrac{14}{2} + 12 + \ldots + 2)$$

Z_4 (min.) $= -\ 140{\cdot}3$ tons.

Z_6.

(Loading boundary in the 7th bay.)

$$V = 4{\cdot}2 \qquad H = 64{\cdot}8$$

$$0 = Z_6 \times 1{\cdot}707 - 4{\cdot}2 \times 10 + 64{\cdot}8 \times 0{\cdot}5$$
$$+ 2{\cdot}4\,(\tfrac{10}{2} + 8 + \ldots + 2) + 4 \times 2$$

Z_6 (max.) $= -\ 34{\cdot}21$ tons.

$$V = 4{\cdot}2 \qquad H = 111{\cdot}2$$

$$0 = Z_6 \times 1{\cdot}707 + 4{\cdot}2 \times 10 + 111{\cdot}2 \times 0{\cdot}5$$
$$+ 2{\cdot}4\,(\tfrac{10}{2} + \ldots + 2) + 4\,(\tfrac{10}{2} + \ldots + 4)$$

Z_6 (min.) $= -\ 146{\cdot}2$ tons.

Z_7.

(Loading boundary in the 8th bay.)

$$V = 5{\cdot}6 \qquad H = 70{\cdot}4$$

$$0 = Z_7 \times 1{\cdot}28 - 5{\cdot}6 \times 8 + 70{\cdot}4 \times 0{\cdot}5$$
$$+ 2{\cdot}4\,(\tfrac{8}{2} + 6 + 4 + 2) + 4 \times 2$$

Z_7 (max.) $= -\ 28{\cdot}74$ tons.

$$V = 5{\cdot}6 \qquad H = 105{\cdot}6$$

$$0 = Z_7 \times 1{\cdot}28 + 5{\cdot}6 \times 8 + 105{\cdot}6 \times 0{\cdot}5$$
$$+ 2{\cdot}4\,(\tfrac{8}{2} + 6 + 4 + 2) + 4\,(\tfrac{8}{2} + 6 + 4)$$

Z_7 (min.) $= -\ 149{\cdot}9$ tons.

Z_8.

(Loading boundary in the 8th bay.)

$$V = 5{\cdot}6 \qquad H = 70{\cdot}4$$

$$0 = Z_8 \times 0{\cdot}943 - 5{\cdot}6 \times 6 + 70{\cdot}4 \times 0{\cdot}5 + 2{\cdot}4\,(\tfrac{6}{2} + 4 + 2)$$

Z_8 (max.) $= -\ 24{\cdot}6$ tons.

$$V = 5{\cdot}6 \qquad H = 105{\cdot}6$$

$$0 = Z_8 \times 0{\cdot}943 + 5{\cdot}6 \times 6 + 105{\cdot}6 \times 0{\cdot}5$$
$$+ 2{\cdot}4\,(\tfrac{6}{2} + 4 + 2) + 4\,(\tfrac{6}{2} + 4 + 2)$$

Z_8 (min.) $= -\ 152{\cdot}8$ tons.

Z_9.

(Loading boundary in the 9th bay.)

$$V = 7{\cdot}2 \qquad H = 76{\cdot}8$$

$$0 = Z_9 \times 0{\cdot}698 - 7{\cdot}2 \times 4 + 76{\cdot}8 \times 0{\cdot}5 + 2{\cdot}4\left(\tfrac{4}{2} + 2\right)$$

$$Z_9\text{ (max.)} = -\,27{\cdot}5 \text{ tons.}$$

$$V = 7{\cdot}2 \qquad H = 99{\cdot}2$$

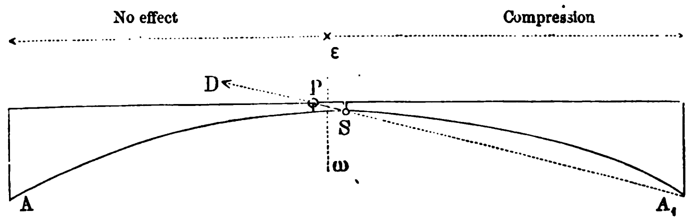
$$0 = Z_9 \times 0{\cdot}698 + 7{\cdot}2 \times 4 + 99{\cdot}2 \times 0{\cdot}5 + 2{\cdot}4\left(\tfrac{4}{2} + 2\right) + 4\left(\tfrac{4}{2} + 2\right)$$

$$Z_9\text{ (min.)} = -\,149{\cdot}0 \text{ tons.}$$

Z_{10}.

A slight alteration occurs in the grouping of the loads in this case, for in no position does the moving load produce tension in Z_{10}. This is shown in Fig. 195. The reason is that the prolongation of the line A_1 S happens to coincide with the diagonal of the 10th bay, but every load to the left of the section line $\epsilon\,\omega$ acts indirectly on the part S $\epsilon\,\omega$ by means of its hinge-reaction D, which passes through the turning point P, and consequently produces no stress in Z_{10}.

FIG. 195.



Thus to find Z_{10} (max.) the bridge can be considered unloaded (or if one chooses, loaded up to the point P), and when Z_{10} (min.) obtains the moving load will cover the bridge (or else up to the section line $\epsilon\,\omega$ only). Hence the following equations:

$$V = 0 \qquad H = 48$$

$$0 = Z_{10} \times 0{\cdot}5498 + 48 \times 0{\cdot}5 + 2{\cdot}4 \times \tfrac{2}{2}$$

$$Z_{10}\text{ (max.)} = -\,48{\cdot}02 \text{ tons.}$$

$$V = 0 \qquad H = 128$$

$$0 = Z_{10} \times 0{\cdot}5498 + 128 \times 0{\cdot}5 + 2{\cdot}4 \times \tfrac{2}{2} + 4 \times \tfrac{2}{2}$$

$$Z_{10}\text{ (min.)} = -\,128{\cdot}05 \text{ tons.}$$

The results obtained are collected together in Fig. 196.

If the signs of all the stresses in Fig. 196 be changed, the stresses in the suspension bridge formed by turning the arch upside down will be obtained, if the abutment hinges become the points of attachment. This suspension bridge is shown in Fig. 197.

§ 23.—Stability of the Abutments of the Braced Arch.

The stability of the abutments can be tested by the method of moments, and it also can be ascertained which distribution of the moving load acts the most injuriously in this respect. The force tending to overturn the abutments or piers is the horizontal component of the thrust of the arch. The vertical component of the same force, on the contrary, adds to the stability. Both components are greatest when the bridge is fully loaded, yet the excess of the moment of the horizontal component over that of the vertical component may reach its maximum with a partial load.

To decide this point the first step is to find the position which a load must occupy on the bridge, so that it may have no overturning effect on the pier. The axis about which the pier tends to rotate is represented in Fig. 198 by the point F,* and for the load Q to have no overturning effect the reaction produced by it at the abutment A_1 must pass through F. Evidently the vertical drawn through the intersection of F A_1 and A S produced gives the required position of the load Q, and it is also easily seen that the reaction for all loads to the right of Q will pass inside F, and for all loads to the left of Q outside F. The worst case for the pier is therefore when the bridge is loaded from the left abutment up to the vertical through I. The position of this vertical evidently depends on the ratio $\frac{h}{b}$ of the height of the pier (up to the hinge A_1) to its breadth.

As an example suppose that

$$\frac{h}{b} = 2,$$

* To allow for the compressibility of the material of which the pier is built, moments should not be taken round F but round an axis nearer the centre of the pier. See Appendix.—T

Fig. 196.

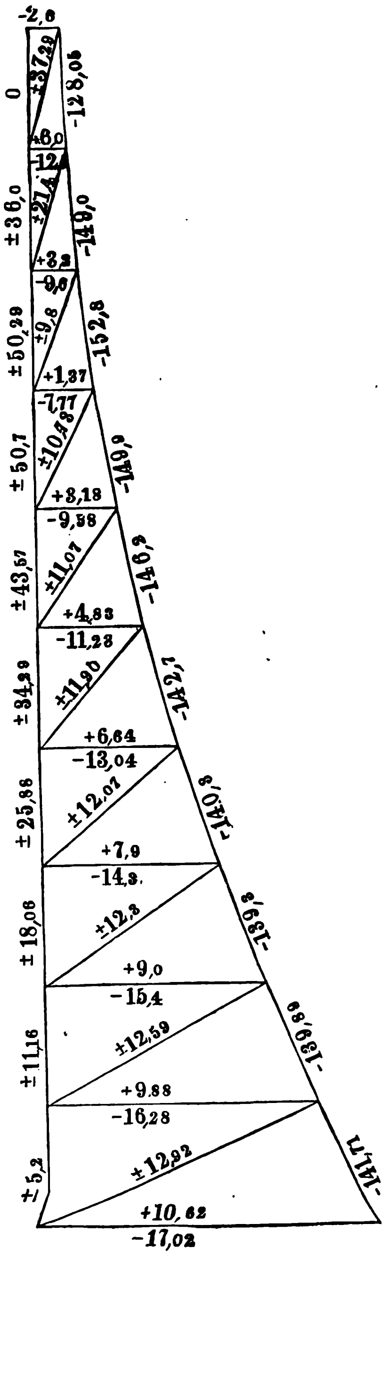
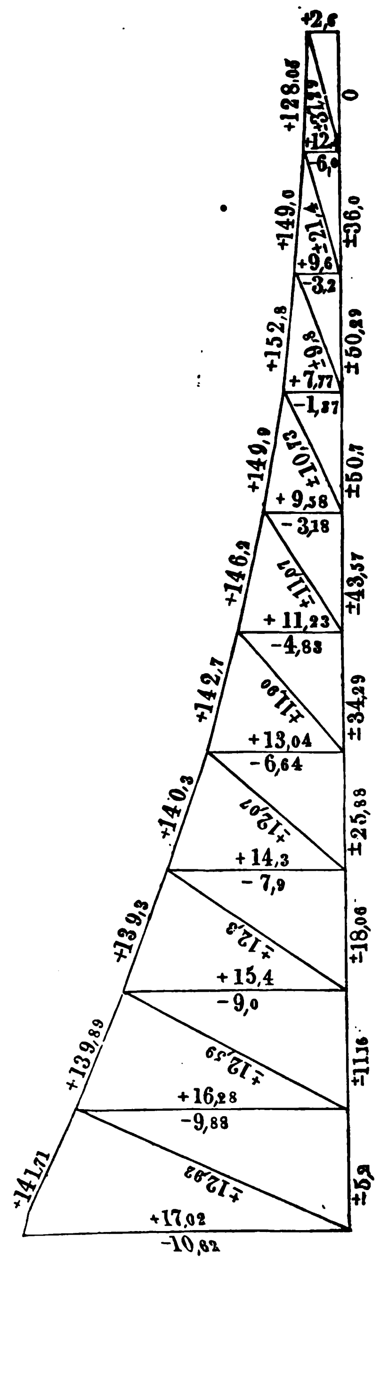


Fig. 197.



then for the arch calculated in the preceding paragraph the loading boundary I Q falls in the 18th bay. Fig. 199 represents the most unfavourable arrangement of the load as regards the pier at A_1. From this figure the equations to obtain the hinge-reaction are:

$$0 = -V \times 20 + H \times 5 - 2{\cdot}4\left(\tfrac{20}{2} + 18 + \ldots + 2\right) - 4\left(\tfrac{20}{2} + 18 + \ldots + 6\right)$$

$$0 = -V \times 20 - H \times 5 + 2{\cdot}4\left(\tfrac{20}{2} + 18 + \ldots + 2\right) + 4\left(\tfrac{20}{2} + 18 + \ldots + 6\right)$$

$$V = 0{\cdot}6 \qquad H = 125{\cdot}6.$$

FIG. 198.

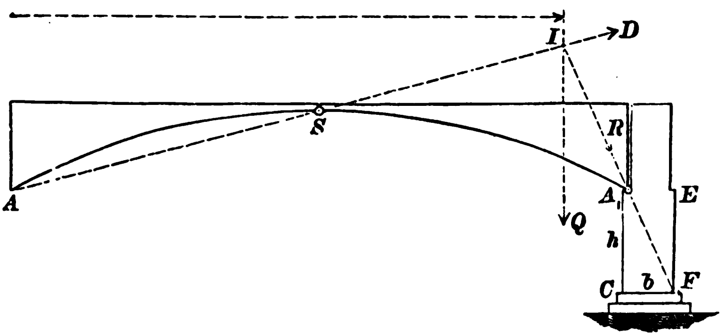
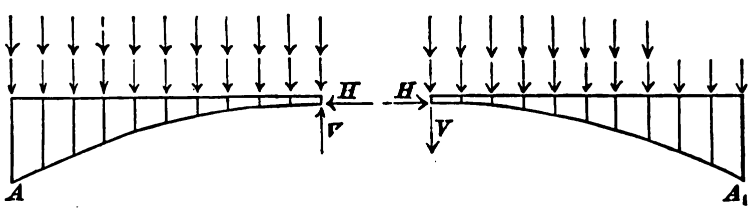


FIG. 199.



Let M represent the overturning moment of the *whole bridge* about the horizontal axis through F (Fig. 198), then:

$$\frac{M}{2} = -0{\cdot}6(b + 20) + 125{\cdot}6(h + 5)$$
$$- 2{\cdot}4\left[\left(\frac{20 + b}{2}\right) + (18 + b) + (16 + b) + \ldots + (2 + b) + \frac{b}{2}\right]$$
$$- 4\left[\left(\frac{20 + b}{2}\right) + (18 + b) + (16 + b) + \ldots + (6 + b)\right].$$

[illegible] the condition of equality

[illegible] of M and M_1 the [illegible] ascertained.*

* [illegible] is the distance between any of the [illegible] A similar case is discussed at p. [illegible]—

§ 24—Theory of Hinged-bridges.

It is now proposed to consider from a more general point of view the principles of construction of these bridges.

The stresses in any system of bars can be calculated by the method of moments as soon as the direction and magnitude of the reactions at the abutments are known. In girder-bridges the abutments are so arranged that they can only produce vertical reactions, and there can therefore be no uncertainty as to their magnitude. But in the case of arched or suspension bridges a horizontal reaction is added to the vertical reaction, and it is only when this former can be determined that the stresses can be calculated.

This horizontal reaction is indeterminate unless the continuity of the structure is interrupted at some point and a hinge introduced, as will be proved by the following.

In the "Theory of parabolic girders" (§ 8) it was shown that the parabola is the curve of equilibrium of an inverted chain in the form of an arch, when the load is uniformly distributed over the span; and in this case both the horizontal and vertical reactions are determinable. But the slightest alteration either in the distribution of the load or in the form of the curve, would make the chain collapse unless it is stiffened by some means. This stiffening can be obtained in two different ways: either by transforming the flexible chain into a stiff bow which prevents deformation by its resistance to flexure, or else by means of a system of braces composed of horizontal, vertical, and diagonal bars, forming triangles with each other. In both cases the flexible arch will be transformed into a stiff structure, and the abutments will have to supply horizontal as well as vertical reactions.

The magnitude of the vertical reactions can always be determined; this will appear by taking moments about the abutment B (Fig. 201 or Fig. 202), thus:

$$0 = V \cdot 2l - Qx,$$

or

$$V = Q \frac{x}{2l}.$$

But the horizontal reactions are indeterminate, for the only condition of equilibrium to which they are subject is that the horizontal force acting at A shall be equal and opposite to that acting at B, and this condition can evidently be satisfied by an infinite number of values. This condition can also be expressed as

FIG. 201.

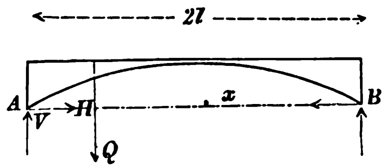
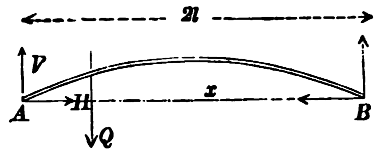


FIG. 202.



follows: the resultants D and W of the reactions at the abutments must, for equilibrium, meet the vertical through Q at the same point. The position of this point on the vertical is, however, indeterminate, and depends on the magnitude of H. The point P (Fig. 203) will lie above the horizontal through the abutments when the horizontal reactions act inwards, and below the horizontal (Fig. 204) when the horizontal reactions act outwards. The nearer P is to the horizontal the greater

FIG. 203.

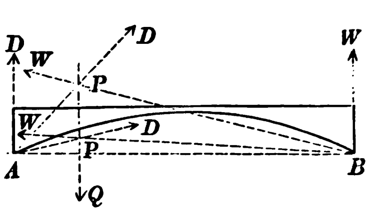
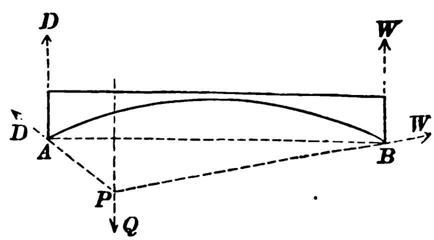


FIG. 204.



the horizontal reactions, and when this distance vanishes these reactions become infinite.

[NOTE.—In reality H is not indeterminate, as will be evident by the application of Canon Moseley's Principle of Least Resistance, which, as stated by Professor Rankine, is as follows:—

"If the forces, which balance each other in or upon a given body or structure be distinguished into two systems, called respectively active and *passive, which* stand to each other in the relation of cause and effect, then will

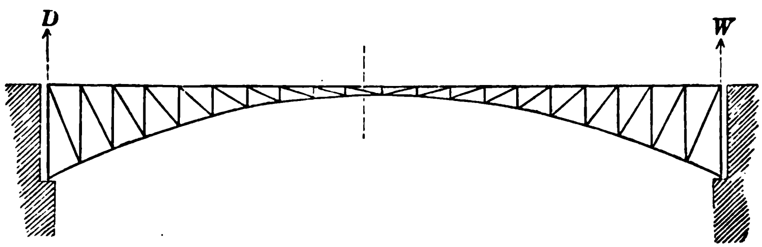
the passive forces be the least which are capable of balancing the active forces, consistently with the physical condition of the body or structure."

Now in the present case the vertical passive forces are determinate, and the principle therefore applies only to H, the constant horizontal thrust in the arch. He must therefore have the least value consistent with stability].

The actual magnitude of this horizontal reaction depends on the attachments, on the resistance of the abutments, on the changes of temperature, and in fact on several causes which can hardly be allowed for by calculation.* Yet this is of the very greatest importance, for the structure could fail either by the horizontal reaction decreasing or increasing considerably.

Suppose, for instance, that the braced arch just calculated were constructed without a hinge in the centre, and that by the abutments giving way slightly, the horizontal reactions vanished, the structure would then become an ordinary girder

FIG. 205.



(Fig. 205), and the stress in the bar X_{10} for instance, could be found according to the previous method, by means of the equation of moments:

$$0 = X_{10} \times 0{\cdot}5 + 6{\cdot}4\left[\left(\tfrac{1}{20} + \tfrac{2}{20} + \ldots + \tfrac{10}{20}\right) 20 \right.$$
$$\left. + \left(\tfrac{11}{20} . 20 - 2\right) + \left(\tfrac{12}{20} . 20 - 4\right) + \ldots + \left(\tfrac{19}{20} . 20 - 18\right)\right],$$

or

$$X_1 \text{ (min.)} = -1280 \text{ tons.}$$

In the braced arch it was found that $X_{10} = \pm 50{\cdot}7$ tons.

* Professor Rankine shows, both in his 'Applied Mechanics' and in his 'Civil Engineering,' how braced arches *without hinges* are to be treated, allowing for the yielding of the abutments, temperature, &c.—TRANS.

Similarly for Z_{10}:

$$0 = Z_{10} \times 0 \cdot 5498 + 6 \cdot 4 \left[\left(\tfrac{1}{20} + \tfrac{2}{20} + \ldots + \tfrac{11}{20} \right) 18 + \left(\tfrac{12}{20} \cdot 18 - 2 \right) + \left(\tfrac{13}{20} \cdot 18 - 4 \right) + \ldots + \left(\tfrac{19}{20} \cdot 18 - 16 \right) \right],$$

or

$$Z_{10} \text{ (max.)} = + 1229 \text{ tons};$$

whereas for the arch:

$$Z_{10} \text{ (min.)} = - 128 \cdot 05 \text{ tons.}$$

Thus, if the horizontal reactions vanish the stress in Z_{10} would be increased nearly ten times, and that in X_{10} more than twenty times.

If, however, the abutments remain firm, and the arch expands

Fig. 206.

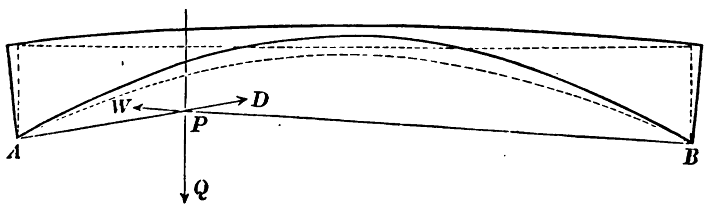
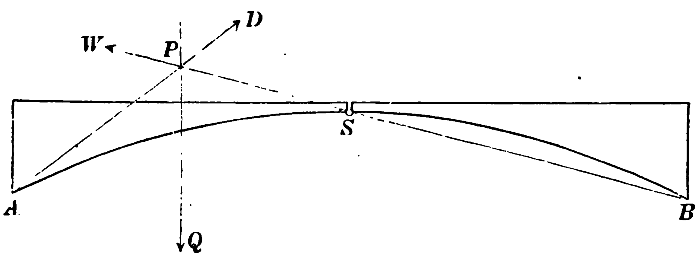


Fig. 207.



by an increase of temperature, the horizontal reaction will be increased, and it is not difficult to see that in some parts the stresses will become considerably greater.

As soon, however, as a hinge is introduced, as at S (Fig. 207), all the indeterminateness as to the magnitude of the horizontal reaction disappears, and likewise the danger caused thereby. It has already been shown that a load Q (Fig. 207) placed on one half of the arch calls forth a reaction at the

abutment of the other half, which must of necessity pass through the central hinge. Consequently by producing B S the position of P can be fixed, and with it the horizontal reactions at the abutments. If the abutments give way slightly, the hinge will be slightly lowered, and it will rise a little when the arch lengthens with an increase of temperature, but in no case will the difference produced in the stresses be appreciable.

It has already been pointed out in § 8, and again at the end of § 20, that there is no difference between the calculations for an arch, that is when the convexity of the bow is turned upwards, and those for a suspension bridge in which the convexity is turned downwards. Thus Fig. 208 is obtained from Fig. 207 by turning the arch upside down, and then

FIG. 208.

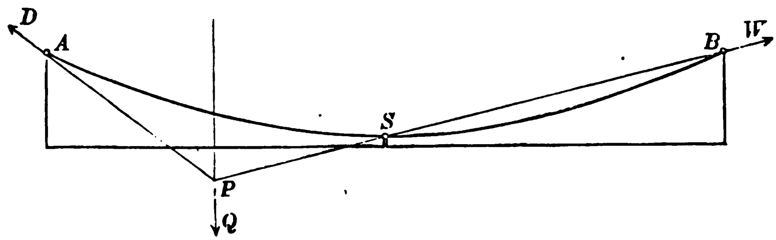


FIG. 209.

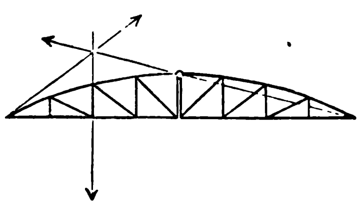
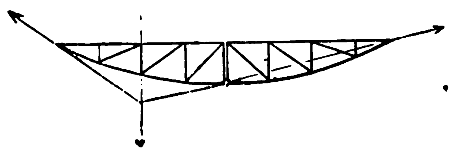


FIG. 210.



changing the direction of all the forces. It is also evident that all the remarks made relatively to the arch also apply to the suspension bridge.

It is hardly necessary to observe that hinge-bridges can be constructed of a variety of forms. Two of these are represented in Figs. 209 and 210. Fig. 209 can be regarded as the para-

bolic girder of Fig. 39, the lower boom of which has been cut through in the centre, and the resistance to tension of the boom replaced by the horizontal reactions at the abutments. Fig. 210 is a similar form in which the resistance to tension of the abutments is brought into requisition. Both structures can be calculated in the manner explained in § 21 and § 22.

SEVENTH CHAPTER.

§ 25.—Variation in the Stresses due to Alterations in the Span.

In the preceding chapters the equations of moments, &c., have been given *in extenso* for each part of the structure, for it is possible to employ these equations and the stresses obtained in many ways for structures that are geometrically similar to those that have been calculated, or as it may be expressed, for structures that differ only in *their* unit of length.

Were it not that the loads alter according to the span, and especially that the proportion between the permanent and moving load changes, the equations and stresses found would be directly applicable whatever the span. For it makes no difference in the results whether the unit of length is a foot, or a metre, or a yard, since the equations of moments depend only on the proportion between the lever arms and not on their absolute length.

If then, when the span increased, the permanent and the moving load increased in the same ratio, it would only be necessary to multiply the stresses already found by this ratio to obtain the new stresses. But in general this cannot be done, for the permanent load as a rule augments much more rapidly as the span increases than the moving load, and consequently an increase of span will affect the stresses in different parts of a structure differently. The problem is therefore to find these new stresses from those already calculated, and to do so by as short a way as possible.

The following notation will be used: p and m will represent the permanent and moving loads on the structure that has already been calculated, and p_1 and m_1 the permanent and moving loads on the new structure.

Now every stress can be divided into two parts, one produced by the permanent and the other by the moving load. If the stress already found be thus divided, and the first part be multiplied by $\frac{p_1}{p}$ the second by $\frac{m_1}{m}$ and the results added the required stress will evidently be obtained.

The various bars of a structure divide themselves into three groups, with respect to the effect of the permanent and moving loads upon them as follows:—

The first group contains all those bars the stress in which depends entirely on the moving load. In this case the new stress is obtained by multiplying the old stress by the ratio $\frac{m_1}{m}$.

The second group comprises those bars in which the stress is greatest when the structure is fully loaded. For them the new stress is equal to the old stress multiplied by $\frac{p_1 + m_1}{p + m}$.

And the third group consists of all the remaining bars, that is those who obtain their greatest stress with a partial load. In this case the stress produced by the permanent load must be multiplied by $\frac{p_1}{p}$ and that by the moving load by $\frac{m_1}{m}$ and the results added together to obtain the new stress.

The last group is the only one which ever requires new calculations, and as a rule these calculations are very simple. The stresses in the bars of the first and second groups can be obtained without difficulty from the stresses already found. As an illustration a few examples are appended—

a. *Parabolic Girder.*

Here the diagonals belong to the first, the horizontal bars to the second, and the verticals to the third group.

Thus to find, from Fig. 27, the stresses in the diagonals of a similar girder, 48 metres span, with a permanent load of 8000 kilos. and a moving load of 12,000 kilos. on each joint, the stress in each of the six diagonals must be multiplied by

$$\frac{12000}{5000} = 2 \cdot 4,$$

and the new stresses are

$$\pm 15000, \quad \pm 16400, \quad \pm 17000, \quad \pm 16400, \quad \pm 15000, \quad \pm 13130.$$

The stress in the horizontal bars must be multiplied by

$$\frac{8000 + 12000}{1000 + 5000} = \frac{10}{3}$$

to find the new stress, which is

$$-\ 160{,}000 \text{ kilos.}$$

Similarly the stresses in the bow are to be multiplied by $\frac{10}{3}$ thus:

$$+\ 175000, \quad +\ 167700, \quad +\ 163000, \quad +\ 160300.$$

The stresses in the verticals must be divided into two parts as explained above. Now the permanent load produces a stress of $-$ 1000 kilos. in each vertical, and therefore the effect of the moving load can be found by adding $-$ 1000 to the stresses given in Fig. 27; thus

For the Maxima Stresses,

$$\begin{Bmatrix} -1000 \\ 0 \end{Bmatrix} \quad \begin{Bmatrix} -1000 \\ +1560 \end{Bmatrix} \quad \begin{Bmatrix} -1000 \\ +2500 \end{Bmatrix} \quad \begin{Bmatrix} -1000 \\ +2800 \end{Bmatrix} \begin{matrix} \text{Permanent load.} \\ \text{Moving load.} \end{matrix}$$

And for the Minima Stresses,

$$\begin{Bmatrix} -1000 \\ -5000 \end{Bmatrix} \quad \begin{Bmatrix} -1000 \\ -6560 \end{Bmatrix} \quad \begin{Bmatrix} -1000 \\ -7500 \end{Bmatrix} \quad \begin{Bmatrix} -1000 \\ -7800 \end{Bmatrix} \begin{matrix} \text{Permanent load.} \\ \text{Moving load.} \end{matrix}$$

and the new stresses are obtained from these by multiplying the first figures in brackets by 8 and the second by 2·4, thus

For the Maxima Stresses,

$$\begin{Bmatrix} -8000 \\ 0 \end{Bmatrix} = -\ 8000 \qquad \begin{Bmatrix} -8000 \\ +3740 \end{Bmatrix} = -\ 4260$$

$$\begin{Bmatrix} -8000 \\ +7000 \end{Bmatrix} = -\ 2000 \qquad \begin{Bmatrix} -8000 \\ +6720 \end{Bmatrix} = -\ 1280$$

And for the Minima Stresses,

$$\begin{Bmatrix} -\ 8000 \\ -12000 \end{Bmatrix} = -\ 20000 \qquad \begin{Bmatrix} -\ 8000 \\ -15700 \end{Bmatrix} = -\ 23700$$

$$\begin{Bmatrix} -\ 8000 \\ -18000 \end{Bmatrix} = -\ 26000 \qquad \begin{Bmatrix} -\ 8000 \\ -18700 \end{Bmatrix} \quad -\ 26700$$

Since the stresses in the remaining three verticals are repetitions of the above, it is unnecessary to calculate them.

b. *Braced Girder with Parallel Booms.*

Here the first group does not occur; all the horizontal bars belong to the second group, and the diagonals and verticals to the third group.

As an example, let it be required to deduce from Fig. 57 the stresses in a similar girder of 48 metres, and (as in the last example) with a permanent load of 8000 kilos. and a moving load of 12000 kilos. on each joint.

The stresses in the horizontal bars are to be multiplied by

$$\frac{8000 + 12000}{1000 + 5000} = \frac{10}{3}$$

to obtain the new stresses—thus,

70000, 120000, 150000, 160000,

which for the lower boom must be taken with a positive sign and for the upper boom with a negative sign.

The stresses in the diagonals and verticals could be as quickly calculated by introducing into the equations of moments given at p. 39–43, the new values of the loads as by the present method of dividing the stresses into two parts. This latter course will, however, be adopted, for in so doing the stress produced by the permanent load alone and by the moving load alone will be found, and thereby a better view of the functions of these braces will be obtained. As an example, take the diagonal and the vertical of the third bay. In § 10 it was found that

$$Y_3 = V_3 \,.\, \sqrt{2},$$

and that

$$0 = Y_3 \times 0 \cdot 707 - 1000\left[\left(\tfrac{1}{8} + \tfrac{2}{8} + \ldots + \tfrac{5}{8}\right) - \left(1 - \tfrac{6}{8}\right) - \left(1 - \tfrac{7}{8}\right)\right]$$
$$- 5000\left(\tfrac{1}{8} + \tfrac{2}{8} + \ldots + \tfrac{5}{8}\right) + 5000\left[\left(1 - \tfrac{6}{8}\right) + \left(1 - \tfrac{7}{8}\right)\right].$$

To find the stress produced in this diagonal by the permanent load alone, leave out the two members multiplied by 5000 and solve the equation, thus obtaining + 2,120 kilos.

Then leave out the permanent load, and the stresses due to the moving load alone are found to be

$$+ 13260 \quad \text{and} \quad - 2650.$$

The last two stresses are to be multiplied by

$$\frac{12000}{5000} = 2 \cdot 4,$$

and the first by

$$\frac{8000}{1000} = 8,$$

and the results added together, thus:

$$\left\{ \begin{matrix} + 16960 \\ + 31824 \end{matrix} \right\} = + 48784 \qquad \left\{ \begin{matrix} + 16960 \\ - 6360 \end{matrix} \right\} = + 10600.$$

The stresses in the vertical V_3 can be found by multiplying the above stresses by $-\frac{1}{\sqrt{2}}$, thus:

$$\left\{ \begin{matrix} - 12000 \\ - 22500 \end{matrix} \right\} = - 33500 \qquad \left\{ \begin{matrix} - 12000 \\ + 4500 \end{matrix} \right\} = - 7500.$$

c. *Braced Arch and Suspension Bridge.*

The diagonal and horizontal bars in this case belong to the first group, the verticals and the bars in the bow to the third group, the second group has no representatives.

As an example let it be required to find the stresses in a suspension bridge geometrically similar to that given in Fig. 197, and having a span of 120 metres.

The permanent load on each joint will be taken at 20 tons, and the moving load at 12 tons. The stresses in the horizontal bars (Fig. 197) are therefore to be multiplied by $\frac{12}{4} = 3$ to obtain the new stresses, thus:

$$\pm 15 \cdot 6, \quad \pm 33 \cdot 48, \quad \pm 54 \cdot 18, \quad \pm 77 \cdot 64, \quad \pm 102 \cdot 87, \quad \pm 130 \cdot 71,$$
$$\pm 152 \cdot 1, \quad \pm 150 \cdot 87, \quad \pm 108 \cdot 0, \quad 0.$$

Likewise the stresses in the diagonals are to be multiplied by 3, thus

$$\pm 38 \cdot 76, \quad \pm 37 \cdot 77, \quad \pm 36 \cdot 9, \quad \pm 36 \cdot 21, \quad \pm 35 \cdot 7 \quad \pm 33 \cdot 21,$$
$$\pm 32 \cdot 19, \quad \pm 29 \cdot 4, \quad \pm 64 \cdot 2 \quad \pm 111 \cdot 87.$$

For the verticals the values of u_1, u_2 already found (p. 124), and representing the effect of the moving load alone (taken with contrary signs for a suspension bridge) can be used.

These values multiplied by the ratio $\frac{12}{4} = 3$ give for the maxima stresses,

$$+ 47\cdot46, \quad + 45\cdot24, \quad + 42\cdot6, \quad + 39\cdot3, \quad + 35\cdot52, \quad + 30\cdot09,$$
$$+ 25\cdot14, \quad + 19\cdot71, \quad + 25\cdot2, \quad + 33\cdot6, \quad + 6;$$

and for the minima stresses

$$- 35\cdot46, \quad - 33\cdot24, \quad - 30\cdot6, \quad - 27\cdot3, \quad - 23\cdot52, \quad - 18\cdot09, \quad - 13\cdot14,$$
$$- 7\cdot71, \quad - 13\cdot2, \quad - 21\cdot6, \quad 0.$$

The stress in the verticals produced by the permanent load is (with the exception of that in the eleventh vertical, which has only one-half to bear) $+ 1\cdot2$ ton, and to obtain the stress in the larger bridge due to the permanent load alone, this must be multiplied by $\frac{20}{2\cdot4} = 8\cdot33$, and the new stress is $1\cdot2 \times 8\cdot33 = 10$ tons, which must be added to the stresses due to the moving load, thus:

+ 57·46	+ 55·24	+ 52·6	+ 49·3	+ 45·52	+ 40·09	+ 35·14
− 25·46	− 23·24	− 20·6	− 17·3	− 13·52	− 8·09	− 3·14

+ 29·71	+ 35·2	+ 43·6	+ 11
− 2·29	− 3·2	− 11·6	+ 5

The stresses in the chains can also be determined by splitting up the stresses as above, for the stress produced by the permanent load alone, which is uniformly distributed over the horizontal span, can be easily found from the formula given in § 8, the chains being in the form of a parabola; the stress due to the moving load alone can then be found by subtraction from the total stress. The result of thus splitting up the stresses given in Fig. 197 is the following, where the upper figures are due to the permanent, and the lower to the moving load.

53·2	52·2	51·2	50·5	49·8	49·3	48·7	48·4	48·1	48
88·5	87·7	88·1	89·8	92·9	96·9	101·2	104·4	100·9	80

The first are to be multiplied by $\frac{20}{2 \cdot 4} = 8 \cdot 33$, and the second by $\frac{12}{4} = 3$, thus:

443	434·5	427	421	415	410	406	403	401	400
265	263	264	269	279	291	304	313	303	240

and the new stresses are obtained by adding these together, thus:

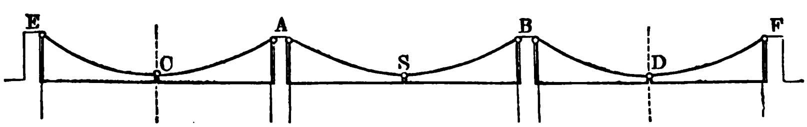
708	697·5	691	690	694	701	710	716	704	640

EIGHTH CHAPTER.

§ 26.—Suspension Bridge in Three Spans. Span of Central Opening, 120 Metres. Span of each Side Opening, 60 Metres.

Suspension bridges do not, unless special arrangements are made, compare favourably with braced arches, as regards the amount of materials employed; for in the latter the points of connection with the abutments are placed low down, and the horizontal thrust acts against the abutments in the direction in which they are strongest; whereas in the former, on the contrary, the points of attachment are placed high up, and the horizontal pull tends to turn the piers over in the direction in which they are weakest; consequently, the quantity of material in the piers will be much greater in one case than in the other. Whereas, therefore, with a braced arch a comparatively small expenditure of material is required for the abutments, especially if natural ones of rock can be obtained, the quantity would be enormous with a suspension bridge, if it were wished to attach the chains to the piers, as shown at E and F (Fig. 211).

Fig. 211.

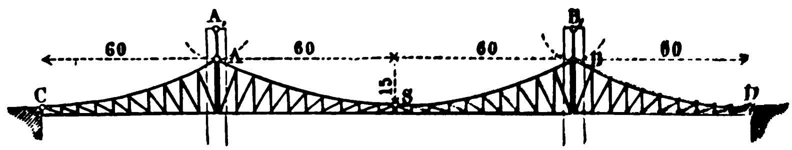


The comparison would, however, be less unfavourable to the suspension bridge if there were several spans, as shown in Fig. 211. The horizontal tensions neutralize each other at the central piers A and B, at least when the spans are equally loaded; but there would be the same disadvantage at the land-

piers E and F.* In such a structure the horizontal and diagonal bars would be under no stress when the bridge is uniformly loaded, assuming the curves to be parabolas.

There would be no alteration, as regards this last point, if the ends E C and D F were cut off and the chains attached at the points C and D to abutments. This arrangement has the advantage of lowering the points of attachment of the chains at the shore end, thereby increasing the stability of the abutments. If, besides this, the points A and B are hung to A_1 and B_1 by means of vertical rods, the central piers will be entirely relieved of all horizontal thrust, even when the load is not uniformly distributed, for the reactions at A_1 and B_1 must of necessity be entirely vertical (Fig. 212). The chains in the

Fig. 212.



parts C A and B D act as land-ties to the central opening, and at the same time the material in them is employed to bridge over the side spans.

(The connections at the points A, B, C, and D, shown in Fig. 212, are only given by way of illustration, other and better means of arriving at the same result will be discussed farther on.)

Such a bridge can, on the whole, be represented by the combination of four rods shown in Fig. 213. These rods are connected together by smooth hinges; they are supported directly by the fixed points C and D, and by means of vertical rods at A and B. It is also supposed that the rods are weightless.

Now it is evident that the direction of a force acting at each end of an unloaded rod must be that of the rod itself; for otherwise rotation would ensue. Therefore a load Q placed at P can only produce a reaction R at C acting in the direction A C; and, similarly, a load Q_1 at P_1 can only give rise to

* It will be observed that it is usual to place land-ties at E and F. This would greatly diminish the quantity of material in the end piers.—Trans.

a reaction in the same direction. But if a load Q be placed anywhere on the rod A C (Fig. 214), the reaction at C will be vertical. For if in this case there were a horizontal thrust at C, an equal horizontal thrust would be required at A and S. But there can be no force acting at S; for since both rods A S and B S are unloaded, this force would be required to have

FIG. 213.

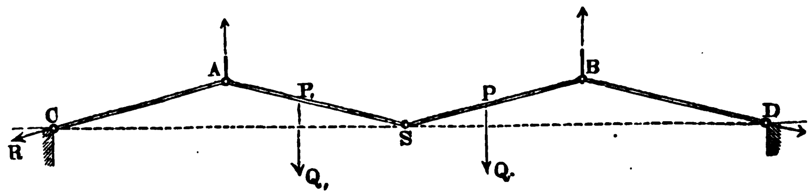
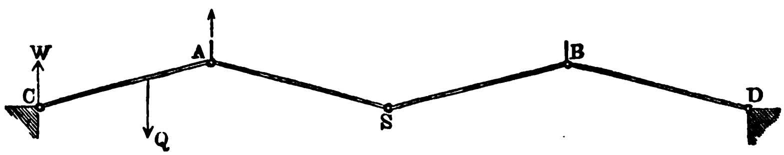


FIG. 214.



simultaneously the directions A S and B S. Thus, a load placed on the rod C A has no effect on the remaining three rods A S B D.

When the rod C A, therefore, is alone loaded, it behaves like an ordinary beam supported at both ends, and when the rods A S, S B are loaded they are in the same condition as if their points of suspension A and B were fixed points.

The stresses in the bridge shown in Fig. 212 can now be found.

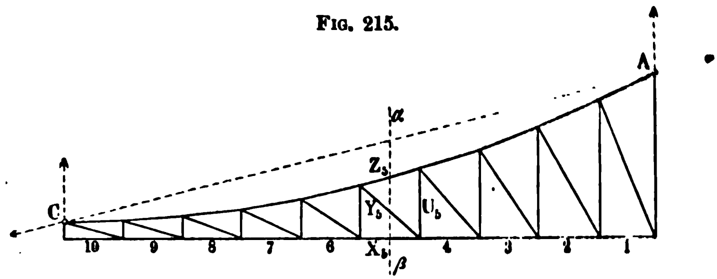
a. *Calculation of the Stresses in the Central Span A B.*

The stresses in the bars of each half of the central span A B can, in accordance with the above, since A and B may be regarded as fixed points, be found by the method employed to calculate those given in Fig. 197. The span is 120 metres, and, assuming that the form is geometrically similar and the loads the same, the stresses found for the suspension bridge in § 25 c, will be those required.

b. *Calculation of the Stresses in the Side Span A C, Fig. 215.*

It will be assumed that the parts A C and B D of the bridge are, as regards their form and construction, geometrically similar to each half of the suspension bridge of § 22, and

FIG. 215.



the same letters have been used to denote the corresponding parts. The loads will also be taken the same as those given in § 25 c, namely, 20 tons permanent,* and 12 tons moving load on each joint. The method adopted for the calculations of the braced arch of § 22 will be followed, and for each bar it will be found which loads produce tension and compression respectively. To do this, recourse must be had to the two laws given above, which are:

1. A load on the central span requires a reaction H at the points A and C, whose direction is A C.

2. A load on the side span A C produces vertical reactions at the points A and B.

Calculation of the Stresses X in the Horizontal Bars.

The stress in X_5 is to be found by taking a section α β, and then forming the equation of moments, either for the part in Fig. 216 or the part in Fig. 217, with reference to the point J. Any load on Fig. 216 produces a vertical reaction W at A, which tends to turn Fig. 217 from right to left round J.

* This assumes that the weight of the bridge is uniformly distributed, and this is not far from the truth, as will appear by examining Figs. [illegible] and [illegible], p. 181.—TRANS.

X_5 acts in the contrary direction, and is therefore positive. The loads on the part $C\,\alpha\,\beta$ belong, therefore, to the tension group.

A load placed on Fig. 217 produces a vertical reaction D at C, making X_5 positive, as before; consequently the loads on $A\,\alpha\,\beta$ also belong to the tension group.

FIG. 216.

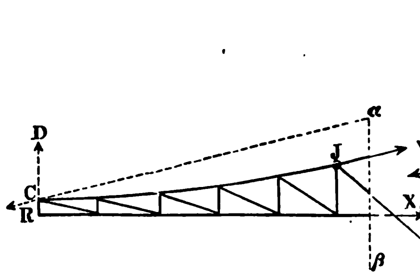
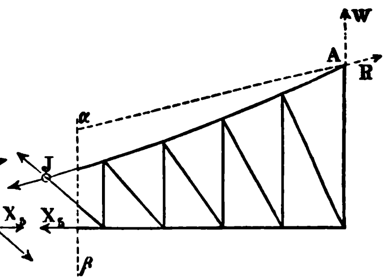


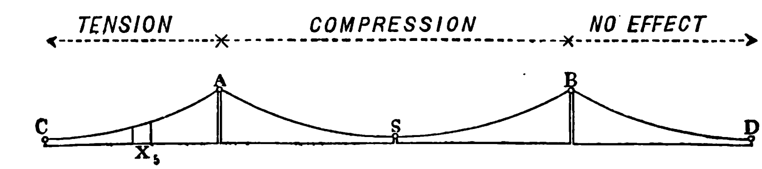
FIG. 217.



A load on the principal span produces a reaction R at C, which tends to turn the part in Fig. 216 from right to left, thus making X_5 negative; therefore the loads on the principal span belong to the compression group.

A load placed in any position on B D has no effect on A C, and the part B D is marked accordingly in Fig. 218.

FIG. 218.

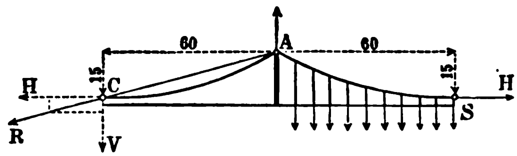


Since when the load is uniformly distributed over the *whole* bridge the stress in the horizontal bars is nothing, the permanent load can be omitted from the calculations; and, further, the maxima and minima values of the stress produced in the horizontal bars by the moving load are numerically equal; therefore it is only necessary to find one of them. The *central span* must alone be loaded when X_5 (min.) obtains, and

the consequent horizontal tension at S (Fig. 219) is given by the equation:

$$0 = -\mathrm{H} \times 15 + 12\left(\tfrac{60}{2} + 54 + 48 + \ldots + 12 + 6\right)$$
$$\mathrm{H} = 240.$$

FIG. 219.



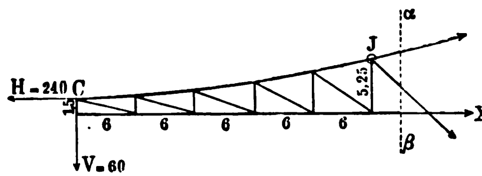
The horizontal component of R must evidently be equal to H, and since the ratio of the vertical to the horizontal component of R is as 15 : 60, or as 1 : 4;

$$\mathrm{V} = 60.$$

Hence the equation of moments from Fig. 220 is

$$0 = -\mathrm{X}_5 \times 5{\cdot}25 + 240 \times 3{\cdot}75 - 60 \times 30$$
$$\mathrm{X}_5 \text{ (min.)} = -171{\cdot}4 \text{ tons.}$$

FIG. 220.



As a check, X_5 (max.) can be computed by taking the side span A C alone loaded, and considering it as a girder, thus:

$$0 = -\mathrm{X}_5 \times 5{\cdot}25 + 12\left[\left(\tfrac{1}{10} + \ldots + \tfrac{5}{10}\right)30 + \left(\tfrac{6}{10} \,.\, 30 - 6\right) + \left(\tfrac{7}{10} \,.\, 30 - 12\right) + \left(\tfrac{8}{10} \,.\, 30 - 18\right) + \left(\tfrac{9}{10} \,.\, 30 - 24\right)\right]$$
$$\mathrm{X}_5 \text{ (max.)} = +171{\cdot}4 \text{ tons,}$$

which agrees exactly.

In a similar manner the following equations are obtained for the remaining horizontal bars:

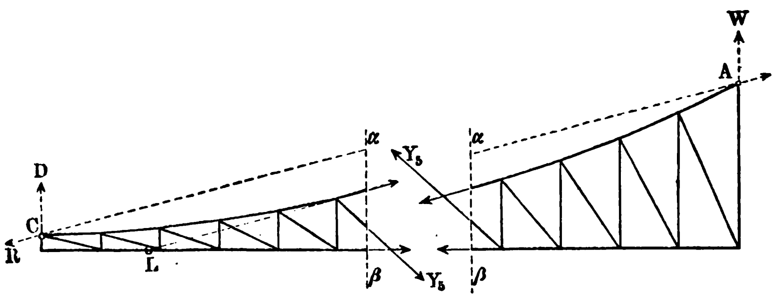
$$0 = -\mathrm{X}_1 \times 13{\cdot}65 + 240 \times 12{\cdot}15 - 60 \times 54$$
$$\mathrm{X}_1 = \pm 23{\cdot}7 \text{ tons}$$

$$0 = -X_2 \times 11{\cdot}1 + 240 \times 9{\cdot}6 - 60 \times 48$$
$$X_2 = \pm 51{\cdot}9 \text{ tons}$$
$$0 = -X_3 \times 8{\cdot}85 + 240 \times 7{\cdot}35 - 60 \times 42$$
$$X_3 = \pm 85{\cdot}4 \text{ tons}$$
$$0 = -X_4 \times 6{\cdot}9 + 240 \times 5{\cdot}4 - 60 \times 36$$
$$X_4 = \pm 125{\cdot}2 \text{ tons}$$
$$0 = -X_6 \times 3{\cdot}9 + 240 \times 2{\cdot}4 - 60 \times 24$$
$$X_6 = \pm 221{\cdot}5 \text{ tons}$$
$$0 = -X_7 \times 2{\cdot}85 + 240 \times 1{\cdot}35 - 60 \times 18$$
$$X_7 = \pm 265{\cdot}3 \text{ tons}$$
$$0 = -X_8 \times 2{\cdot}1 + 240 \times 0{\cdot}6 - 60 \times 12$$
$$X_8 = \pm 274{\cdot}3 \text{ tons}$$
$$0 = -X_9 \times 1{\cdot}65 + 240 \times 0{\cdot}15 - 60 \times 6$$
$$X_9 = \pm 196{\cdot}4 \text{ tons}$$
$$0 = -X_{10} \times 1{\cdot}5$$
$$X_{10} = 0.$$

Calculation of the Stresses Y in the Diagonals.

As in the preceding case, and for the same reason, it is unnecessary to consider the permanent load, and it is only requisite to calculate either the maximum or the minimum

FIG. 221. FIG. 222.

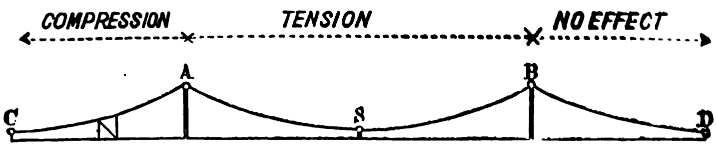


value of the stress. The stress in Y_5, for instance, can be found by means of the equation of moments formed either for the part shown in Fig. 221, or for that shown in Fig. 222, with respect to the point L.

The reaction R at C, due to a load on the central span,

evidently makes Y_5 positive. A load placed anywhere on the part $A\,\alpha\,\beta$ produces a vertical reaction D at the point C, which makes Y_5 negative. A load on $C\,\alpha\,\beta$ requires a vertical reaction W at A, which also makes Y_5 negative. Hence Fig. 223, showing the manner in which the stress in Y_5 is affected by the various loads.

FIG. 223.



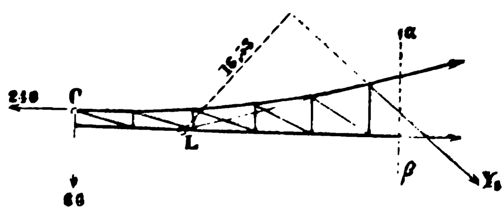
Thus to find Y_5 (max.) the central span alone must be loaded, and, as before (p. 169),

$$H = 240 \quad \text{and} \quad V = 60;$$

and from Fig. 224 the equation of moments is

$$0 = Y_5 \times 16{\cdot}53 - 240 \times 1{\cdot}5 - 60 \times 10{\cdot}02$$
$$Y_5 = \pm\, 61{\cdot}4 \text{ tons.}$$

FIG. 224.



Similarly, for the remaining diagonals:

$$0 = Y_1 \times 31{\cdot}8 - 240 \times 1{\cdot}5 - 60 \times 25{\cdot}26$$
$$Y_1 = \pm\, 59{\cdot}0 \text{ tons}$$
$$0 = Y_2 \times 28{\cdot}28 - 240 \times 1{\cdot}5 - 60 \times 21{\cdot}88$$
$$Y_2 = \pm\, 59{\cdot}2 \text{ tons}$$
$$0 = Y_3 \times 24{\cdot}48 - 240 \times 1{\cdot}5 - 60 \times 18{\cdot}39$$
$$Y_3 = \pm\, 59{\cdot}8 \text{ tons}$$
$$0 = Y_4 \times 20{\cdot}5 - 240 \times 1{\cdot}5 - 60 \times 14{\cdot}77$$
$$Y_4 = \pm\, 60{\cdot}8 \text{ tons}$$

$$0 = Y_6 \times 12 \cdot 72 - 240 \times 1 \cdot 5 - 60 \times 6 \cdot 67$$
$$Y_6 = \pm 59 \cdot 7 \text{ tons}$$
$$0 = Y_7 \times 9 \cdot 58 - 240 \times 1 \cdot 5 - 60 \times 1 \cdot 713$$
$$Y_7 = \pm 48 \cdot 3 \text{ tons.}$$

In the case of Y_8 the turning-point is situated to the left of C, consequently the sign of the moment of the vertical reaction at D is reversed, and thus the loads producing tension extend over the central span and up to the section line $\gamma\,\delta$, as shown in Fig. 225. In this case it is easier to calculate Y_8 (min.), and from Fig. 226, or else from Fig. 227, the following equation of moments is obtained:

$$0 = - Y_8 \times 7 \cdot 53 - 12 \left(\tfrac{1}{10} + \tfrac{2}{10}\right) 64 \cdot 8$$
$$Y_8 = \pm 31 \text{ tons.}$$

FIG. 225.

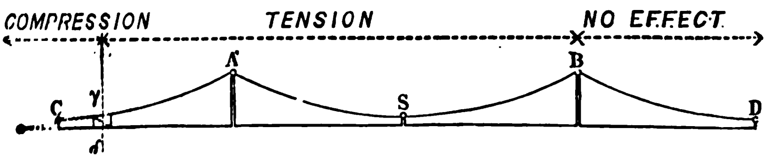
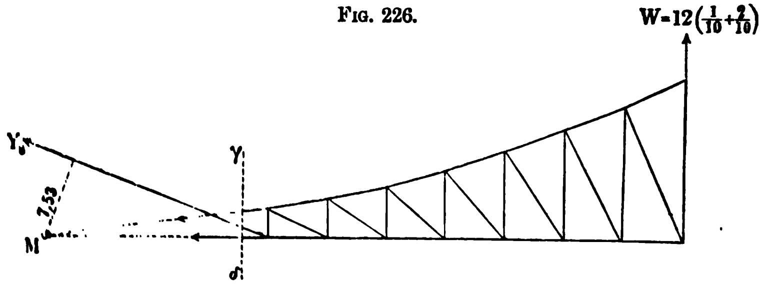


FIG. 226.



The turning-point for Y_9 is situated so far to the left that the moment of R, the reaction due to the loads on the central span, also changes its sign, and consequently the loads on this span produce compression in Y_9 (Fig. 228). Here again it is easiest to calculate Y_9 (min.). Thus from Fig. 229:

$$0 = Y_9 \times 7 \cdot 41 - 240 \times 1 \cdot 5 + 60 \times 16 + 12 \left(22 - \tfrac{3}{10} \times 16\right)$$
$$Y_9 = \pm 93 \cdot 3 \text{ tons.}$$

Lastly, for Y_{10} all the loads on A C produce tension, and Y_{10} can therefore be obtained by considering the central span loaded, and as before (p. 171):

$$0 = Y_{10} \times 15{\cdot}97 - 240 \times 1{\cdot}5 + 60 \times 60$$

$$Y_{10} = \pm 202{\cdot}9 \text{ tons.}$$

Fig. 227.

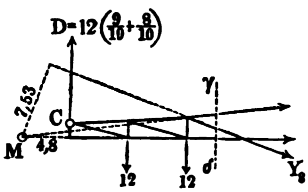


Fig. 229.

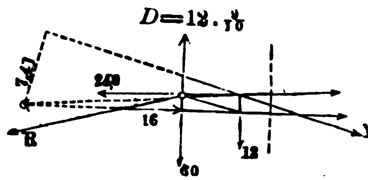
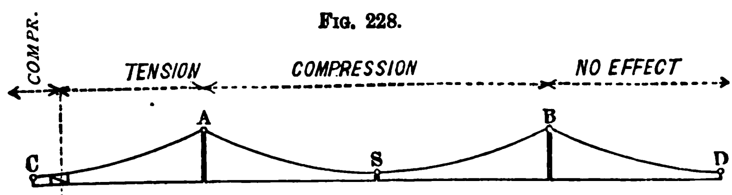


Fig. 228.



Calculation of the Stresses U in the Verticals.

The stress in each vertical can be divided into two parts; one part due to the permanent load, and the other to the moving load. The first is the same for all the verticals, and is equal to + 10 tons, if it is assumed that one-half of the total permanent load (20 tons) is applied to the upper joints, and the other half to the lower joints. Denoting the part of the stress due to the moving load by u, it is evident that

$$u\,(\text{max.}) + u\,(\text{min.}) = +\,12 \text{ tons,}$$

because when the moving load covers the bridge, it produces a tension of 12 tons in each vertical (being applied to the lower joints only). Thus, if u (min.) be calculated, u (max.) can be found from the equation

$$u\,(\text{max.}) = +\,12 - u\,(\text{min.}).$$

And finally, the total stress in the verticals can be found by means of the equations

$$U\ (\text{max.}) = u\ (\text{max.}) + 10$$
$$U\ (\text{min.}) = u\ (\text{min.}) + 10.$$

The turning-points for the verticals are the same as those for the diagonals; the loading boundaries will therefore in general be the same, but the loads that produce tension in the diagonals will produce compression in the verticals, and *vice versâ.* The bars U_9 and U_{10}, however, possess a second loading boundary, which is determined by the section line itself, as was also found to be the case with the corresponding diagonals, but will be shifted one bay to the left, owing to the oblique direction of the section line.

FIG. 230.

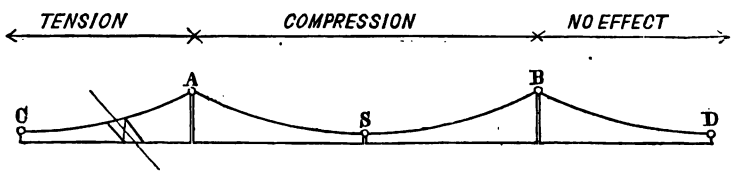
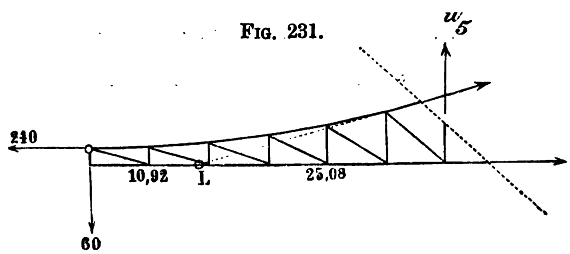


FIG. 231.



The groups of loads for U_5 are shown in Fig. 230; and from Fig. 231 the equation of moments to find u_5 (min.) is

$$0 = -u_5 \times 25 \cdot 08 - 240 \times 1 \cdot 5 - 60 \times 10 \cdot 92$$
$$u_5\ (\text{min.}) = -40 \cdot 5;$$

and substituting in the above equations

$$u_5\ (\text{max.}) = +12 - (-40 \cdot 5) = +52 \cdot 5 \text{ tons}$$
$$U_5\ (\text{max.}) = +52 \cdot 5 + 10 = +62 \cdot 5 \text{ tons}$$
$$U_5\ (\text{min.}) = -40 \cdot 5 + 10 = -30 \cdot 5 \text{ tons.}$$

In the same manner the stresses in the remaining verticals can be found, thus:

$$0 = -u_1 \times 34{\cdot}74 - 240 \times 1{\cdot}5 - 60 \times 25{\cdot}26$$
$$u_1\,(\text{min.}) = -54 \qquad u_1\,(\text{max.}) = +66$$
$$U_1\,(\text{min.}) = -44 \text{ tons} \qquad U_1\,(\text{max.}) = +76 \text{ tons}$$

$$0 = -u_2 \times 32{\cdot}12 - 240 \times 1{\cdot}5 - 60 \times 21{\cdot}88$$
$$u_2\,(\text{min.}) = -52{\cdot}1 \qquad u_2\,(\text{max.}) = +64{\cdot}1$$
$$U_2\,(\text{min.}) = -42{\cdot}1 \text{ tons} \qquad U_2\,(\text{max.}) = +74{\cdot}1 \text{ tons}$$

$$0 = -u_3 \times 29{\cdot}61 - 240 \times 1{\cdot}5 - 60 \times 18{\cdot}39$$
$$u_3\,(\text{min.}) = -49{\cdot}5 \qquad u_3\,(\text{max.}) = +61{\cdot}5$$
$$U_3\,(\text{min.}) = -39{\cdot}5 \text{ tons} \qquad U_3\,(\text{max.}) = +71{\cdot}5 \text{ tons}$$

$$0 = -u_4 \times 27{\cdot}23 - 240 \times 1{\cdot}5 - 60 \times 14{\cdot}77$$
$$u_4\,(\text{min.}) = -45{\cdot}8 \qquad u_4\,(\text{max.}) = +57{\cdot}8$$
$$U_4\,(\text{min.}) = -35{\cdot}8 \text{ tons} \qquad U_4\,(\text{max.}) = +67{\cdot}8 \text{ tons}$$

$$0 = -u_6 \times 23{\cdot}33 - 240 \times 1{\cdot}5 - 60 \times 6{\cdot}67$$
$$u_6\,(\text{min.}) = -32{\cdot}6 \qquad u_6\,(\text{max.}) = +44{\cdot}6$$
$$U_6\,(\text{min.}) = -22{\cdot}6 \text{ tons} \qquad U_6\,(\text{max.}) = +54{\cdot}6 \text{ tons}$$

$$0 = -u_7 \times 22{\cdot}29 - 240 \times 1{\cdot}5 - 60 \times 1{\cdot}713$$
$$u_7\,(\text{min.}) = -20{\cdot}8 \qquad u_7\,(\text{max.}) = +32{\cdot}8$$
$$U_7\,(\text{min.}) = -10{\cdot}8 \text{ tons} \qquad U_7\,(\text{max.}) = +42{\cdot}8 \text{ tons.}$$

For the three following verticals it is best to find the value of u (max.) thus:

$$0 = -u_8 \times 22{\cdot}8 - 12\left(\tfrac{1}{10} + \tfrac{2}{10} + \tfrac{3}{10}\right) 64{\cdot}8$$
$$u_8\,(\text{max.}) = +20{\cdot}5 \qquad u_8\,(\text{min.}) = -8{\cdot}5$$
$$U_8\,(\text{max.}) = +30{\cdot}5 \text{ tons} \qquad U_8\,(\text{min.}) = +1{\cdot}5 \text{ tons}$$

$$0 = -u_9 \times 28 - 240 \times 1{\cdot}5 + 60 \times 16$$
$$+\, 12\left[\left(22 - \tfrac{9}{10}\,.\,16\right) + \left(28 - \tfrac{8}{10}\,.\,16\right)\right]$$
$$u_9\,(\text{max.}) = +31{\cdot}2 \qquad u_9\,(\text{min.}) = -19{\cdot}2$$
$$U_9\,(\text{max.}) = +41{\cdot}2 \text{ tons} \qquad U_9\,(\text{min.}) = -\ 9{\cdot}2 \text{ tons}$$

$$0 = -u_{10} \times 66 - 240 \times 1{\cdot}5 + 60 \times 60 + 12\left(66 - \tfrac{9}{10}\,.\,60\right)$$
$$u_{10}\,(\text{max.}) = +51{\cdot}3 \qquad u_{10}\,(\text{min.}) = -39{\cdot}3$$
$$U_{10}\,(\text{max.}) = +61{\cdot}3 \text{ tons} \qquad U_{10}\,(\text{min.}) = -29{\cdot}3 \text{ tons.}$$

Lastly, the only stress in the 11th vertical is that due to a load hung underneath it. The greatest value this load can have is $\frac{12}{2}$ tons moving load added to $\frac{10}{2}$ tons permanent load; hence

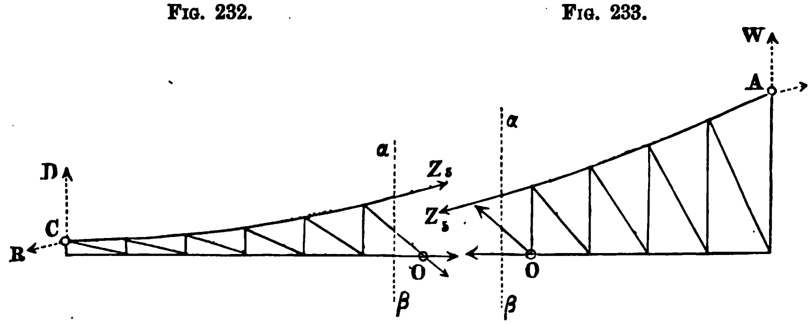
$$U_{11}\,(\text{max.}) = +11 \text{ tons.}$$

Calculation of the stresses Z in the chains.

The sign of Z_5 (Figs. 232 and 233) depends on the sign of the moments of the three forces, R, D, and W. From Fig. 232 it appears that the moments of R (the reaction due to a load on the central span) and Z_5 about O have different signs; therefore a load on the central span makes Z_5 positive.

When the part $A\alpha\beta$ (Fig. 233) is loaded, the reaction D is produced at C, the sign of whose moment about O (Fig. 232) is the same as that of Z_5, or Z_5 is negative.

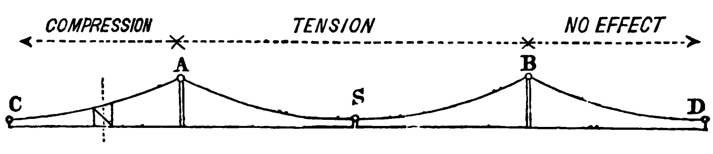
FIG. 232. FIG. 233.



Again, if $C\alpha\beta$ is loaded, the reaction W at A has (Fig. 233) a moment about O, whose sign is the same as that of Z_5; therefore Z_5 is again negative.

The greatest compression occurs, therefore, when the side

FIG. 234.



span is fully loaded, and the greatest tension when the central span is loaded, as shown in Fig. 234.

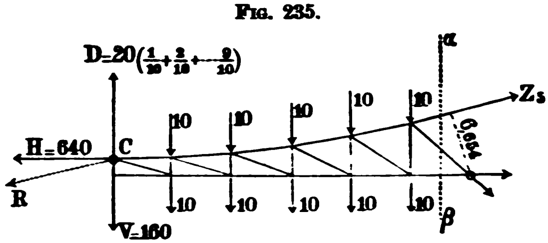
The stress in Z_5 can be calculated in two different ways. The first is as follows:—When Z_5 (max.) obtains, the central span is fully loaded, and the side span has only the perma-

nent load upon it. From Fig. 235 the equation of moments is

$$0 = Z_5 \times 6{\cdot}654 - H \times 1{\cdot}5 - V \times 36 + D \times 36 - 20\,(6 + 12 + \ldots + 30),$$

where H and V are the components of the reaction R, due to the load on the central span. The corresponding values of H and V, due to the moving load alone, have already been found

FIG. 235.



to be 240 and 60 respectively. To find the values now required, these must be multiplied by the ratio

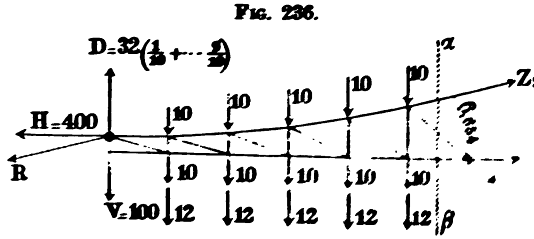
$$\frac{20 + 12}{12} = \frac{8}{3}.$$

Further, D is the vertical reaction at C, due to the permanent load on the side span.

Substituting, the equation of moments becomes

$$0 = Z_5 \times 6{\cdot}654 - 640 \times 1{\cdot}5 - 160 \times 36 + 20\left(\tfrac{1}{10} + \ldots + \tfrac{9}{10}\right)36$$
$$- 20\,(6 + 12 + \ldots + 30)$$
$$Z_5\ (\text{max.}) = +\ 794 \text{ tons.}$$

FIG. 236.



The value of Z_5 (min.) can be obtained from Fig. 236. If H and V are the components of the reaction R, due to the per-

manent load alone on the central span, and D is the vertical reaction produced by the total load on the side span. Hence

$$H = \tfrac{20}{32} \,.\, 640 = 400$$
$$V = \tfrac{20}{32} \,.\, 160 = 100$$

and

$$= Z_5 \times 6{\cdot}654 - 400 \times 1{\cdot}5 - 100 \times 36 + 32\left(\tfrac{1}{10} + \ldots + \tfrac{9}{10}\right) 36$$
$$- 32\,(6 + 12 + \ldots + 30)$$
$$Z_5 \text{ (min.)} = +\ 285 \text{ tons.}$$

The second method is to split up the stress in Z_5 in two parts, one p_5 due to the permanent load alone, and the other z_5 due to the moving load alone. The value of p_5 has already been obtained in § 25 (for when the bridge is covered with a uniform load, the side spans are precisely in the same condition as either of the halves of the main span). It was found that

$$p_5 = +\ 415 \text{ tons.}$$

It is only necessary to calculate either z_5 (max.) or z_5 (min.), for both together must be equal to the stress produced by the moving load when it covers the whole bridge; and this stress can easily be found by comparison with p_5—in fact, by multiplying p_5 by the ratio $\frac{12}{20} = \frac{3}{5}$; therefore

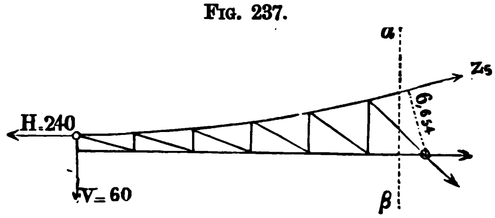
$$z_5 \text{ (max.)} + z_5 \text{ (min.)} = \tfrac{3}{5} \times 415 = +\ 249,$$

or

$$z_5 \text{ (min.)} = +\ 249 - z_5 \text{ (max.)}.$$

It is easiest to obtain z_5 (max.), and it can be found from the equation of moments for the part of the side span shown

FIG. 237.



in Fig. 237 (the values of H and V will be those already found when the moving load covers the central span). Hence

$$0 = z_5 \times 6{\cdot}654 - 240 \times 1{\cdot}5 - 60 \times 36$$
$$z_5 \text{ (max.)} = 379 \text{ tons,}$$

or

$$z_5 \text{ (min.)} = +249 - 379 = -130 \text{ tons.}$$

Finally, adding p_5 and z_5 together,

$$Z_5 \text{ (max.)} = 415 + 379 = +794 \text{ tons}$$
$$Z_5 \text{ (min.)} = 415 - 130 = +285 \text{ tons.}$$

The same result is thus obtained by both methods; the last, however, is the simpler, and will therefore be employed for the calculation of the stresses in the remaining bars Z; thus:—

$$0 = z_1 \times 14{\cdot}904 - 240 \times 1{\cdot}5 - 60 \times 60$$
$$z_1 \text{ (max.)} = +265{\cdot}5$$
$$z_1 \text{ (min.)} = +265{\cdot}5 - 265{\cdot}5 = 0$$
$$Z_1 \text{ (max.)} = 443 + 265{\cdot}5 = +708{\cdot}5 \text{ tons}$$
$$Z_1 \text{ (min.)} = 443 + 0 \quad = +443 \text{ tons}$$

$$0 = z_2 \times 12{\cdot}56 - 240 \times 1{\cdot}5 - 60 \times 54$$
$$z_2 \text{ (max.)} = 286{\cdot}5$$
$$z_2 \text{ (min.)} = 261 - 286{\cdot}5 = -25{\cdot}5$$
$$Z_2 \text{ (max.)} = 434{\cdot}5 + 286{\cdot}5 = +721 \text{ tons}$$
$$Z_2 \text{ (min.)} = 434{\cdot}5 - 25{\cdot}5 = +409 \text{ tons}$$

$$0 = z_3 \times 10{\cdot}39 - 240 \times 1{\cdot}5 - 60 \times 48$$
$$z_3 \text{ (max.)} = 312$$
$$z_3 \text{ (min.)} = 256 - 312 = -56$$
$$Z_3 \text{ (max.)} = 427 + 312 = +739 \text{ tons}$$
$$Z_3 \text{ (min.)} = 427 - 56 = +371 \text{ tons}$$

$$0 = z_4 \times 8{\cdot}415 - 240 \times 1{\cdot}5 - 60 \times 42$$
$$z_4 \text{ (max.)} = 342$$
$$z_4 \text{ (min.)} = 252 - 342 = -90$$
$$Z_4 \text{ (max.)} = 421 + 342 = +763 \text{ tons}$$
$$Z_4 \text{ (min.)} = 421 - 90 = +331 \text{ tons}$$

$$0 = z_6 \times 5{\cdot}121 - 240 \times 1{\cdot}5 - 60 \times 30$$
$$z_6 \text{ (max.)} = 422$$
$$z_6 \text{ (min.)} = 246 - 422 = -176$$
$$Z_6 \text{ (max.)} = 410 + 422 = +832 \text{ tons}$$
$$Z_6 \text{ (min.)} = 410 - 176 = +234 \text{ tons}$$

$$0 = z_7 \times 3{\cdot}84 - 240 \times 1{\cdot}5 - 60 \times 24$$
$$z_7 \text{ (max.)} = 469$$
$$z_7 \text{ (min.)} = 244 - 469 = -225$$
$$Z_7 \text{ (max.)} = 406 + 469 = +875 \text{ tons}$$
$$Z_7 \text{ (min.)} = 406 - 225 = +181 \text{ tons}$$

$$0 = z_8 \times 2{\cdot}83 - 240 \times 1{\cdot}5 - 60 \times 18$$
$$z_8\,(\text{max.}) = 509$$
$$z_8\,(\text{min.}) = 242 - 509 = -267$$
$$Z_8\,(\text{max.}) = 403 + 509 = +912 \text{ tons}$$
$$Z_8\,(\text{min.}) = 403 - 267 = +136 \text{ tons}$$

$$0 = z_9 \times 2{\cdot}094 - 240 \times 1{\cdot}5 - 60 \times 12$$
$$z_9\,(\text{max.}) = 516$$
$$z_9\,(\text{min.}) = 241 - 516 = -275$$
$$Z_9\,(\text{max.}) = 401 + 516 = +917 \text{ tons}$$
$$Z_9\,(\text{min.}) = 401 - 275 = +126 \text{ tons}$$

$$0 = z_{10} \times 1{\cdot}649 - 240 \times 1{\cdot}5 - 60 \times 6$$
$$z_{10}\,(\text{max.}) = 437$$
$$z_{10}\,(\text{min.}) = 240 - 437 = -203$$
$$Z_{10}\,(\text{max.}) = 400 + 437 = +837 \text{ tons}$$
$$Z_{10}\,(\text{min.}) = 400 - 203 = +197 \text{ tons}$$

The results of the above calculations are collected together in Fig. 238. And the stresses in the central span are given in Fig. 239, having been deduced from Fig. 197 and § 21.

§ 27.—Stability of the Central Piers.

It was assumed, in the preceding calculations, that the connections at the points of support were made as indicated in Fig. 212. For these calculations to be true, it is necessary that at the points A and B vertical forces only should act on the bridge, and it therefore follows that these points should be perfectly free to move in a horizontal direction. If such a mode of attachment be adopted, the stability of the central piers is a question that need not be considered (in so far as the vertical forces on the bridge are concerned). The manner of forming these connections shown in Fig. 212 is not, however, the only one by which this advantage may be gained, and it was only chosen as an illustration, and there are better ways of arriving at the same result. For instance, the points A_1 and B_1 can be placed below A and B, as shown in Fig. 240.

Nor is it necessary that the chains of the adjacent spans

Fig. 238.

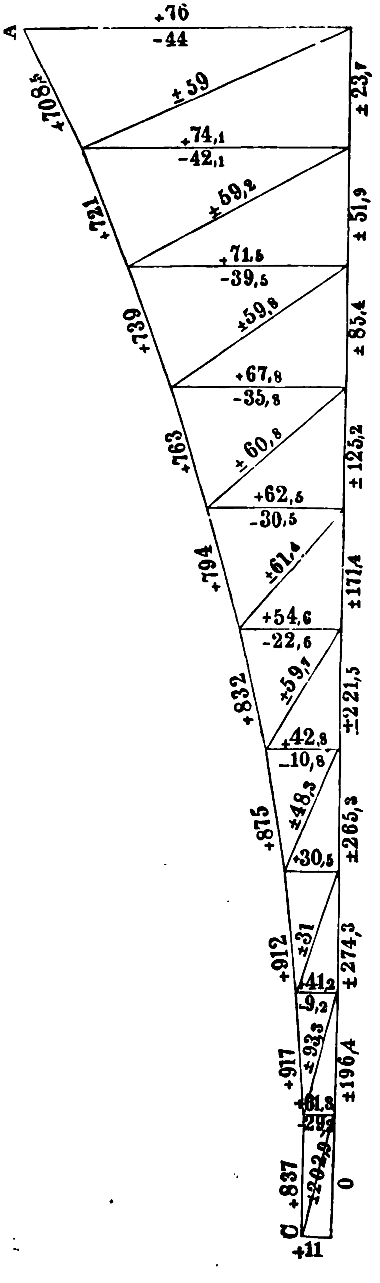
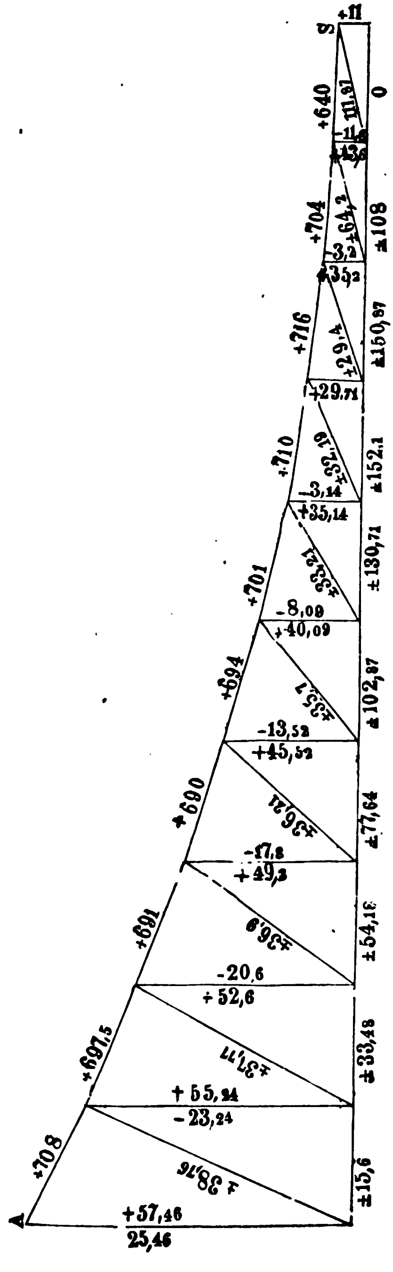


Fig. 239.



should be attached to the same point A; in fact, it is better to place them apart the full width of the pier, and attach them to the points a_1, a_2, Fig. 241. The span is thus slightly diminished. The freedom of these points to move horizontally can be obtained in a variety of ways. Thus, in Fig. 241 an unbraced parallelogram is formed by the three bars $a_1 a_2$, $a_1 b_1$,

FIG. 240.

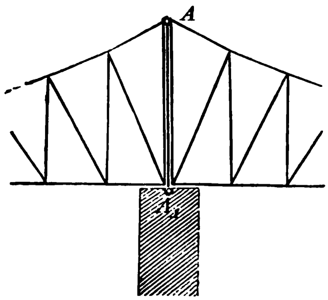
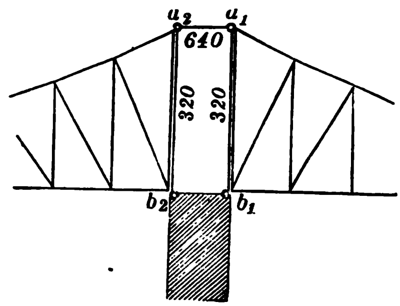


FIG. 241.



and $a_2 b_2$, the fourth side being the head of the pier $b_1 b_2$. (The stresses in these three bars are found immediately from the former calculations, and are inscribed in the figure.)

Or the chains can be attached to the axis of two friction rollers (Fig. 242), and the piers being carried up form the

FIG. 242.

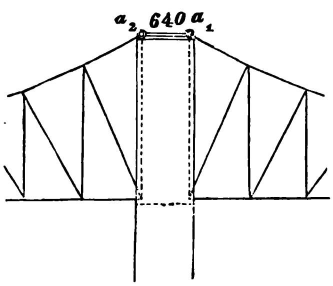
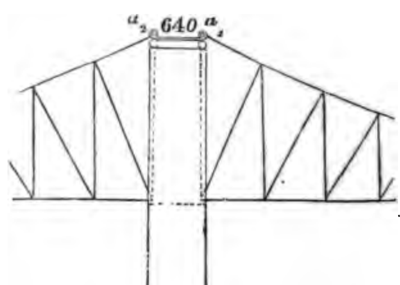


FIG. 243.

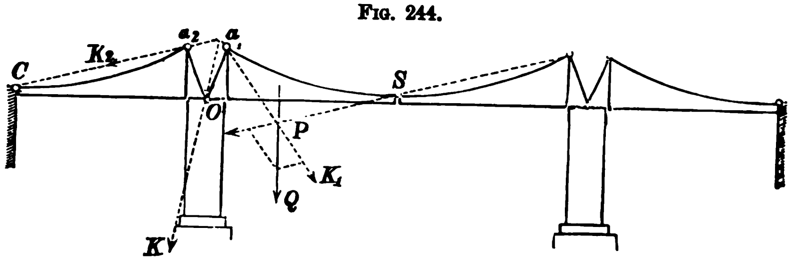


roller path. Or again, the chains may be fastened to a plate placed upon rollers, the pier being carried up as in the former case (Fig. 243).

But if the arrangement shown in Fig. 244 were adopted, the stresses obtained in § 26 would no longer be true, and the advantage of having no lateral thrust on the central piers would

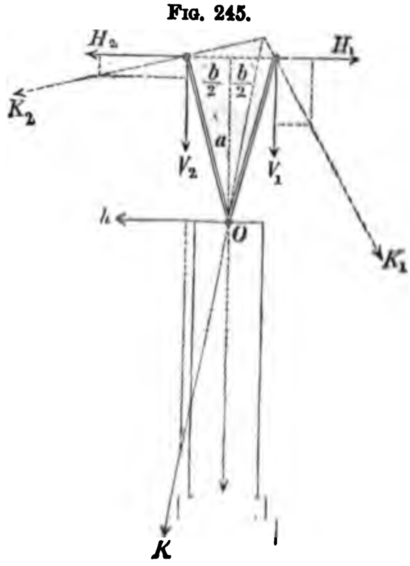
also have to be given up. It is true that when the bridge is fully loaded the reaction at the central piers would be vertical, but this would not be the case with a partial load.

FIG. 244.



This will become apparent by finding the reaction at the fixed point O, due to a single load Q (considering the structure to have no weight). By proceeding as in §§ 22 and 26, it will be found that the force K_1 (Figs. 244 and 245) acts in the direction a_1 P, and the force K_2 in the direction a_2 C. These two forces, together with the reaction at O, maintain the bent lever a_1 O a_2, in equilibrium, and their resultant K must therefore pass through O. The horizontal component h of K is the force that tends to overturn the pier, and will be greatest when all the loads producing the same effect as Q are on

FIG. 245.



the bridge. These loads extend from a_1 to S, for a load situated to the right of S has no overturning effect on the pier in question, since it acts through its hinge-reaction a_1 S, the horizontal component of which is equal to H_2.

In the previous example the moving load from a_1 to S was = 120 tons, and when this load is on the bridge it will be found that

$$H_1 = 120 \quad \text{and} \quad V_1 = 90.$$

Now taking moments about O,

$$0 = H_1 a + V_1 \frac{b}{2} - H_2 a - V_2 \frac{b}{2}.$$

But it was found, p. 169, that

$$V_2 = \tfrac{1}{4} H_2.$$

Hence solving for H_2,

$$H_2 = H_1 \left(\frac{8 + 3\frac{b}{a}}{8 + \frac{b}{a}} \right);$$

but (Fig. 245)

$$h = H_2 - H_1,$$

$$\therefore\ h = 2 H_1 \left(\frac{\frac{b}{a}}{8 + \frac{b}{a}} \right).$$

Putting for H_1 its value (120 tons), and assuming, for example, that $\frac{b}{a} = \frac{1}{2}$,

$$2h = 28 \cdot 2 \text{ tons},$$

or 28·2 tons is the maximum horizontal thrust of the central span of the bridge against either of the central piers.

The horizontal thrust in the contrary direction, produced by loading a_2 C would be very nearly as great.

The force h diminishes together with $\frac{b}{a}$, and becomes nothing when $\frac{b}{a} = 0$; that is, when both arms of the bent lever unite and form a single strut. This is the construction of Fig. 241, and it or one of its modifications is to be preferred.

The vertical pressure on the central piers is, when the bridge is fully loaded, equal to the load on the parts A S and A C together, and is therefore

$$2\,(32 \times 10 + 32 \times 10) = 1280 \text{ tons}.$$

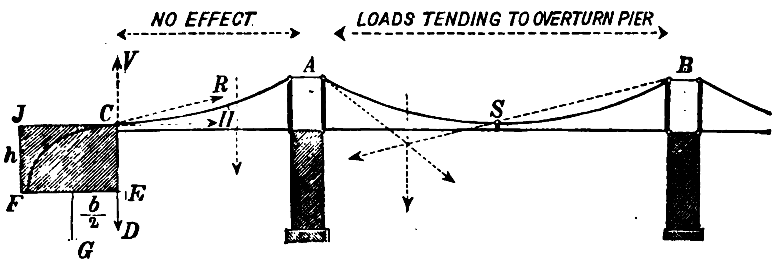
[NOTE.—It is easily seen that this must be the case when it is considered that A C and A S, being equal, will balance about A when equally loaded. Or again, —In Fig. 235 it is shown that V, the vertical component of the pull pro-

duced at C by A S when fully loaded, is 160 tons. When A C is fully loaded, the vertical reaction at C is also 160 tons. The two vertical forces acting at C are therefore equal and opposite, and hence neutralize each other. Thus the whole load on A C and A S is supported at A.]

§ 28.—Stability of the Shore Abutments.

The reaction R at C, due to a load on the central span (Fig. 246), tends to overturn the abutment about its lower edge E, and also to make it slip along its bed F E. Every

Fig. 246.



load on the side span C A produces but a vertical pressure D at C, which is neutral as regards overturning, and helps the abutment to resist sliding.

The moment of the overturning force will thus be greatest when the central span is fully loaded, in which case the horizontal component of R is

$$H\ (\text{max.}) = +\ 640 \text{ tons.}$$

The vertical component of R passes through E, and therefore (similarly to D) is neutral as regards overturning. Thus the condition of stability is that the moment of G (the weight of the abutment) about E is not less than the moment of the horizontal pull, 2 H, of the whole bridge about the same point. This is expressed by

$$G\,\frac{b}{2} > 2 \times 640 \times h, \qquad [1]$$

from which the least dimensions of the pier to resist overturning can be found.

But it must also be ascertained that the pier will not slide.

Both the components of R act injuriously in this respect, H directly, and R indirectly in that it diminishes the pressure on the base, and thereby also the resistance to sliding.

The reaction D, however, increases the resistance to sliding. The danger of sliding will therefore be greatest when the central span is fully loaded and the side span unloaded (moving load), in which case D is equal to half the weight of the part A C (Fig. 247). For these conditions of loading

$$H = 640, \qquad V = 160, \qquad D = 100;$$

and these values must be doubled to represent the effect of the bridge.

Hence, if f is the coefficient of friction,

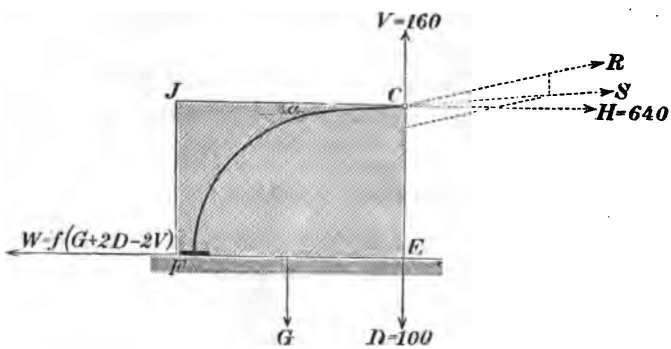
$$f(G + 2D - 2V) > 2H,$$

or

$$f(G + 2 \times 100 - 2 \times 160) > 2 \times 640, \qquad [2]$$

in order that the abutments may not slide.

Fig. 247.



To prevent failure, the value of G must be taken at least as great as the greater of the values obtained from the two conditions expressed in [1] and [2].

The chain C A must be securely attached to the abutment pier, and this can be done by means of a chain built in the masonry (Fig. 247) and anchored at F. The direction of this

chain must not be horizontal at C; for if such were the case, C would rise when the central span was loaded and the side span unloaded. In this case, V = 160 tons and D = 100 tons, and the direction of the chain in the masonry must be such as to supply the vertical force necessary for equilibrium. Thus if a is the angle the chain makes with the horizontal at C, this angle must at least be as great as the angle made with the horizontal by the resultant of R and D (or of the three forces, H, V, and D) when the bridge is loaded as above. The tangent of this last angle is

$$\frac{V - D}{H} = \frac{160 - 100}{640} = \frac{3}{32} = 0 \cdot 09375;$$

therefore

$$\tan a > 0 \cdot 09375,$$

or

$$a > 5^\circ \, 22'.$$

The greatest tension in the chain can be obtained by making its horizontal component equal to the maximum value of H, or 640 tons.

NINTH CHAPTER.

§ 29.—On the Calculation of the Stresses in Domes.

In all the preceding examples it could be assumed that every joint was equally loaded as well as regards the permanent as the moving load. If the weight of the structure itself were not quite uniformly distributed over the span, the difference in each case was small and did not affect to any appreciable extent the values of the stresses found.

But in the case of domes, the error entailed by such an assumption would be too great; for the ribs or principals radiate from the centre like the spokes of a wheel, and consequently the loads on them increase considerably from the centre towards the abutments.

The surface of a dome can be considered as generated by the revolution of a properly shaped curve round the vertical axis, and if the ribs are equally spaced the portion of this surface contained between the vertical planes through two adjacent principals will represent the load on each principal. Further, if the bays formed by the bracing are equal, the loads on each joint will vary as the length of the arc of the circle (seen on plan) passing through the joint and contained between two adjacent ribs. But the length of these arcs is proportional to their distance from the vertical axis of the dome. Thus if p is the load on the joint situated at the unit of distance from the axis, $\rho\, p$ will be the load on a joint placed at a distance ρ from the axis. If, therefore, the load on any joint be known, the load on any other joint can at once be found by simply measuring its horizontal distance from the vertical axis.

Once the loads on the various joints are known, the stresses can be found, and conveniently so, by the method of moments, as will appear in the following example:—

§ 30.—Dome of 100 metres Span.

The exterior surface of the dome is a hemisphere of 51 metres radius, and contains 16,338 square metres. There are

eight ribs, each in the form of a quadrant of a circle, and each rib supports 2042 square metres of the surface of the dome.

The load per square metre of the surface of the dome is assumed to be 235 kilos. (consisting of the weight of the covering, together with that of the snow and wind pressure). Each rib has therefore 2042 × 235 = 480,000 kilos. nearly, or 480 tons (1000 kilos. to the ton) to carry. The whole of this load will be considered variable, not only on account of the snow and wind pressure, but also because it is possible that part of the covering might be removed for a time. The only permanent load is the weight of the rib itself, which is estimated at 60 tons. This load can be considered as equally distributed on the exterior joints. Each rib consists of two concentric booms, 2 metres apart and connected together by triangular bracing, dividing the rib into fifteen bays of equal length (Figs. 248, 249). The permanent load is therefore 4 tons on each exterior joint.

To find the distribution the variable load on the joints, the distance of these joints from the vertical axis must be measured. These distances are as follows:—

DISTANCE—NO. OF JOINT.

0	5·3	10·6	15·8	20·7	25·5	30	34·1	37·9	41·3
S	1	2	3	4	5	6	7	8	9

44·2	46·6	48·5	49·9	50·7	51
10	11	12	13	14	15

These numbers, as already seen, are proportional to the variable load on each joint. Therefore if the whole of the variable load on the rib, 480 tons, be divided by the sum of all these numbers, 512, the quotient multiplied by each number in succession will give the variable load acting on the corresponding joint, thus:

LOAD—NO. OF JOINT.

5	9·9	14·8	19·4	23·9	28·1	32	35·5	38·7	41·4
1	2	3	4	5	6	7	8	9	10

43·7	45·5	46·8	47·6
11	12	13	14

A ring is attached to the feet of the ribs, so that the walls supporting the dome may have no horizontal thrust to bear, and the ribs are exactly in the same condition as if their lower extremities were attached to fixed points.

FIG. 248.

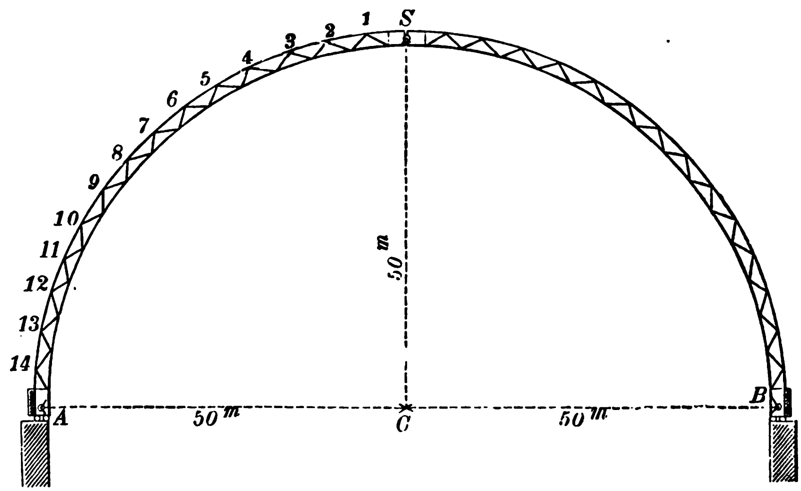
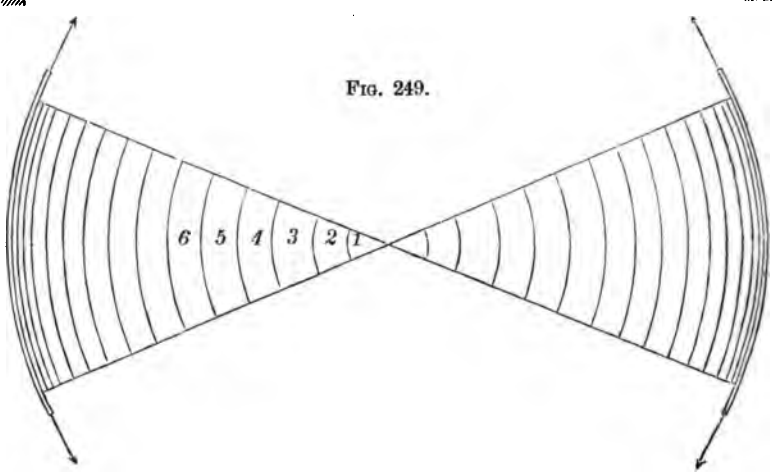


FIG. 249.



Also, in order that the stresses may be independent of the variations of temperature or of alterations in the tension of the

ring, it will be assumed that the tops of the ribs are connected together by a hinge.* Hinges are also placed at A and B.

The arrangement adopted is thus similar to that of the braced arch of § 22, and the reasoning will consequently also be similar.

The process to find the hinge-reaction at S will, for instance, be the same. It will thus be found that when the dome is fully loaded, or when it is quite unloaded, the vertical component of this hinge-reaction is zero, and the reaction is therefore horizontal. The magnitude of this horizontal force H can be found by equating its moment about A to the moment of all the loads about the same point. The lever arms of the loads can be obtained by subtracting their distances from the centre given above from 50, the half radius, thus:

LEVER ARMS—NO. OF JOINT.

50	44·7	39·4	34·2	29·3	24·5	20	15·9	12·1	8·7
S	1	2	3	4	5	6	7	8	9
		5·8	3·4	1·5	0·1	−0·7			
		10	11	12	13	14			

Thus when the ribs are unloaded, H is found from the equation

$$H \times 50 = 4\left(\tfrac{50}{2} + 44{\cdot}7 + 39{\cdot}4 + \ldots + 1{\cdot}5 + 0{\cdot}1 - 0{\cdot}7\right) = 1056,$$

or

$$H = 21{\cdot}12 \text{ tons}.$$

If, however, the full variable load is applied, the equation becomes

$$\begin{aligned} H \times 50 = {} & 4\left(\tfrac{50}{2} + 44{\cdot}7 + 39{\cdot}4 + \ldots + 1{\cdot}5 + 0{\cdot}1 - 0{\cdot}7\right) \\ & + 5 \times 44{\cdot}7 + 9{\cdot}9 \times 39{\cdot}4 + \ldots + 45{\cdot}5 \times 1{\cdot}5 \\ & + 46{\cdot}8 \times 0{\cdot}1 - 47{\cdot}6 \times 0{\cdot}7. \end{aligned}$$

The last products in the equation are the moments of the

* To meet the objection that might be raised, that the number of hinges crossing each other at S would render the construction impossible, the ribs are supposed to abut against a ball, which will act as a hinge for each rib.

several variable loads about A, and as they will occur frequently in the sequel, they are tabulated here:

223·5	390	506·2	568·4	585·6	562	508·8	429·6
1	2	3	4	5	6	7	8
	336·7	240·1	148·6	68·3	4·7	−33·3	
	9	10	11	12	13	14	

The substitution of these values in the equation gives

$$H \times 50 = 1056 + 223{\cdot}5 + \ldots + 4{\cdot}7 - 33{\cdot}3 = 5595,$$

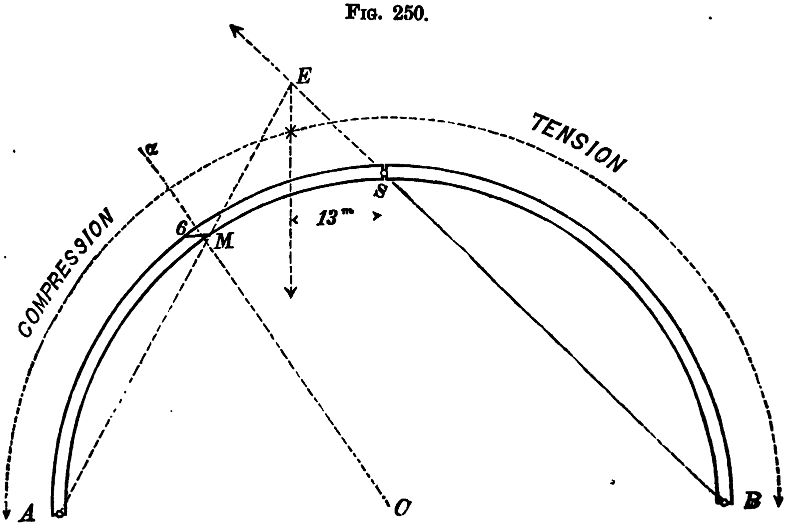
or

$$H = 111{\cdot}9 \text{ tons.}$$

Calculation of the Stresses X in the Outer Boom.

The part of the outer boom situated between joints 5 and 6 will be taken to illustrate the calculations (Fig. 250); M

Fig. 250.



is the turning point, and the loading boundary is therefore found by producing A M and B S to meet at E; then the vertical through E is the required boundary (compare § 22). This

vertical is at a distance of 13 metres from the axis, and falls between the second and third joints, the effect of the various loads is therefore as shown in Fig. 250.

The stress in X will be a maximum when the joints 3, 4, 5, ... 14* are unloaded and the remainder loaded.

But the same result is obtained by considering every joint loaded, and applying to joints 3, 4, 5, ... 14 vertical upward forces equal to their respective variable loads.

To find the components of the hinge-reaction under these conditions, the equations of moments of each rib about its point of support for a full load are to be used, deducting, however, from the equation for the left rib the moments of the twelve unloading forces, thus:

For the right rib,

$$0 = H \times 50 + V \times 50 - 5595$$

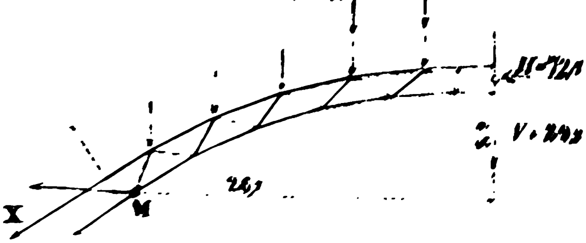
For the left rib,

$$0 = -H \times 50 + V \times 50 + 5595 - 506 - 568 - 586$$
$$\cdots - 4{\cdot}7 + 83{\cdot}3$$
$$H = 72{\cdot}7 \qquad V = 39{\cdot}2.$$

Then from Fig. 251 the equation of moments to obtain X (max.) is:

$$0 = -X \times 2 - 72{\cdot}7 \times 8{\cdot}9 + 39{\cdot}2 \times 26{\cdot}7$$
$$+ 4\left(\frac{26{\cdot}7}{2} + 21{\cdot}4 + 16{\cdot}1 + 10{\cdot}9 + 6 + 12\right)$$
$$+ 5 \times 21{\cdot}4 + 9{\cdot}9 \times 16{\cdot}1$$
$$X\ (\text{max.}) = +\ 470{\cdot}9 \text{ tons.}$$

Fig. 251.



* Since the vertical through the 14th joint passes to the left of the turning point A the load upon it produces tension in X, and the same remark applies to the joint near B, consequently there are in reality four groups of loads, but the error entailed by taking only two groups as shown is so small that it may be neglected.

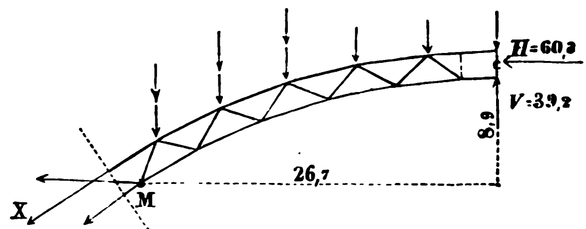
X (min.) obtains when the joints 3, 4, 5, ... 14 alone are loaded, and in this case the components of the hinge-reaction can be found from the following equations of moments:

$$0 = H \times 50 - V \times 50 - 1056$$
$$0 = -H \times 50 - V \times 50 + 1056 + 506 + 568 + 586 + \ldots + 4 \cdot 7 - 33 \cdot 3$$
$$H = 60 \cdot 3 \qquad V = 39 \cdot 2;$$

and from Fig. 252 the equation of moments to find X (min.) is:

$$0 = -X \times 2 - 60 \cdot 3 \times 8 \cdot 9 - 39 \cdot 2 \times 26 \cdot 7 + 4\left(\frac{26 \cdot 7}{2} + 21 \cdot 4 + 16 \cdot 1 + 10 \cdot 9 + 6 + 12\right) + 14 \cdot 8 \times 10 \cdot 9 + 19 \cdot 4 \times 6 + 23 \cdot 9 \times 1 \cdot 2$$
$$X\ (\text{min.}) = -500 \cdot 6 \text{ tons.}$$

FIG. 252.



Calculation of the Stresses Z in the Lower Boom.

As an example, the stress in the part of the lower boom cut through by the section line C *a* (Fig. 250) will be found. The point 6 is the turning point, and the vertical through F, the intersection of A 6 and B S (Fig. 253), is therefore the loading boundary. This vertical is at a distance of 17·3 metres from the axis, and is situated between the third and fourth joints. When, therefore, the stress in Z is a maximum, the joints 4, 5 ... 14 are alone loaded, and the equations to find the hinge-reaction are:

$$0 = H \times 50 - V \times 50 - 1056$$
$$0 = -H \times 50 - V \times 50 + 1056 + 568 + 586 + \ldots + 4 \cdot 7 - 33 \cdot 3$$
$$H = 55 \cdot 3 \qquad V = 34 \cdot 2;$$

and the equation to find Z (max.) is from Fig. 254:

$$0 = Z \times 2 - 55{\cdot}3 \times 8{\cdot}74 - 34{\cdot}2 \times 30$$
$$+ 4\left(\tfrac{32}{2} + 24{\cdot}7 + 19{\cdot}4 + 14{\cdot}2 + 9{\cdot}3 + 4{\cdot}5\right)$$
$$+ 19{\cdot}4 \times 9{\cdot}3 + 23{\cdot}9 \times 4{\cdot}5$$
$$Z\ (\text{max.}) = + 436{\cdot}5 \text{ tons.}$$

FIG. 253.

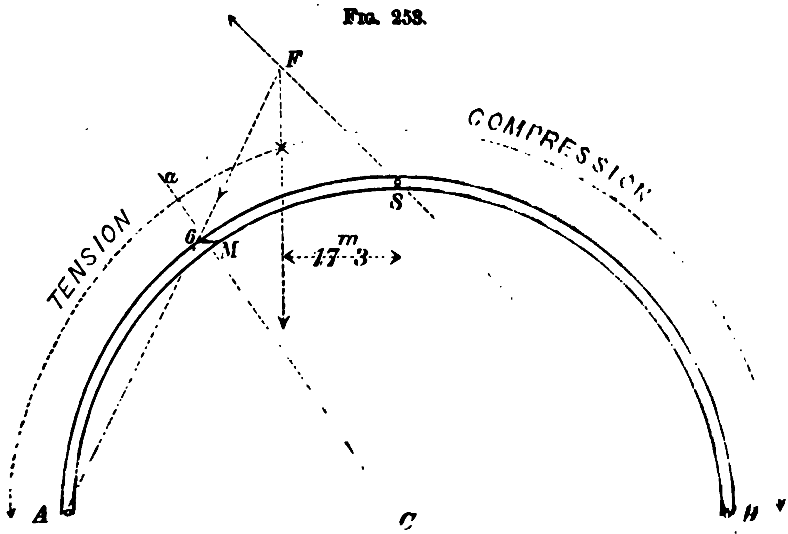
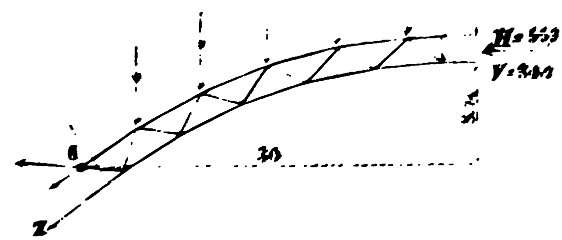


FIG. 254.



When Z (min.) obtains, the joints 4, 5, . . . 14 must be unloaded, and to determine the hinge-reaction the moments due to the loads on these joints must be deducted from the equation of moments of the left rib, considered fully loaded, thus:

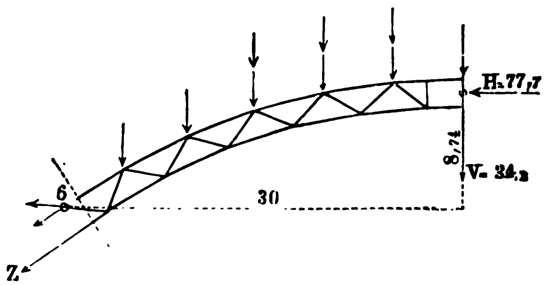
$$0 = H \times 50 - V \times 50 - 5595$$
$$0 = -H \times 50 + V \times 50 + 5650 - 568 - 598 - \ldots - 4{\cdot}7 + 38{\cdot}3;$$
$$H = 77{\cdot}7 \qquad V = 34{\cdot}2.$$

therefore the equation of moments to find Z (min.) obtained from Fig. 255 is:

$$\begin{aligned} 0 = Z \times 2 - 77{\cdot}7 \times 8{\cdot}74 + 34{\cdot}2 \times 30 \\ + 4\left(\tfrac{30}{2} + 24{\cdot}7 + 19{\cdot}4 + 14{\cdot}2 + 9{\cdot}3 + 4{\cdot}5\right) \\ + 5 \times 24{\cdot}7 + 9{\cdot}9 \times 19{\cdot}4 + 14{\cdot}8 \times 14{\cdot}2 \end{aligned}$$

$$Z\ (\text{min.}) = -\ 610{\cdot}5 \text{ tons.}$$

FIG. 255.



Calculation of the Stresses Y in the Diagonals.

Of the two diagonals placed between the joints 9 and 10, the one connected to joint 10 will be chosen to illustrate the method of calculation.

It will be seen that the case that occurred in § 9 is repeated here, namely, that the point about which moments are taken is infinitely distant. The direction of the straight line containing this point is that of the tangent to the circle (centre C) at the point where the diagonal is cut (Fig. 256). The radius at this point makes an angle of $58\frac{1}{2}°$ with the vertical axis, and the tangent therefore also makes an angle of $58\frac{1}{2}°$ with the horizontal. The only difference between this case and that of § 9 is, that in the latter case, the turning point was in the horizontal, and in the present it is in the direction of the tangent.

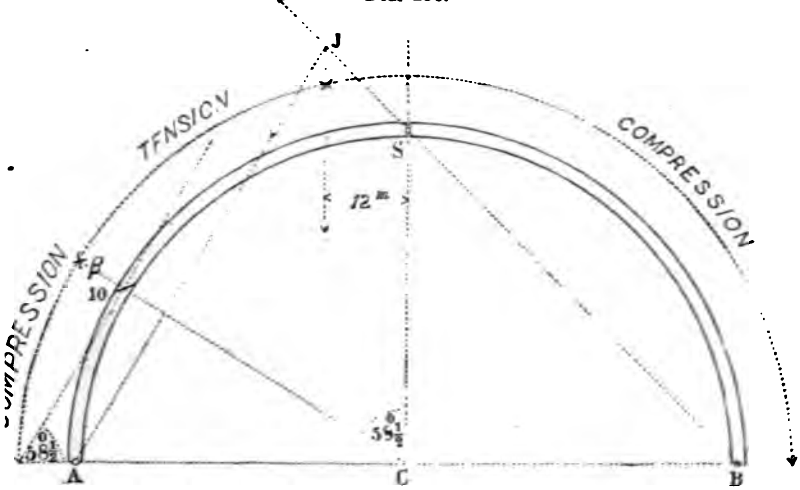
The simplest way is to resolve every force acting on S β into two components, one parallel to the tangent, and the other to the normal at the point where the diagonal is cut; evidently the moments of all the former components is zero.

Let the normal component of Y be denoted by N, then all the loads that make N positive will also make Y positive. *Therefore the* forces that are acting in the same direction as N

make Y negative, and those that are acting in the opposite direction make Y positive.

The loading boundary will thus be seen to be the vertical through the intersection of B S, with a line through A drawn parallel to the tangent. For the direction of the resultant of a load placed in the vertical through J and its hinge-reaction, is J A, and the resultant has therefore no component parallel to the

FIG. 256.



normal. All the loads to the right of the boundary make N negative, and those to the left as far as the section line make N positive. The loads on the other side of the section line, however, again make N negative, for they act indirectly on the portion S β by means of their hinge-reactions. The loads therefore divide themselves into three groups, as shown in Fig. 256.

By construction, it is found that the distance of the vertical through J from the axis is 12 metres, and that the loading boundary falls between the second and third joint. Thus the force N, and therefore also Y, is a maximum when the joints 3, 4, 5, 6, 7, 8, 9 are loaded. With this loading the equations to find the components of the hinge-reaction are:

$$0 = H \times 50 - V \times 50 - 1026$$

$$0 = -H \times 50 - V \times 50 + 1026 + [illegible] + [illegible] + [illegible] + [illegible] + [illegible] + [illegible] + [illegible]$$

$$H = [illegible] \qquad V = [illegible].$$

To find N, the resolved parts parallel to N of all the forces acting on the portion of the rib shown in Fig. 257 must be equated to zero, thus:

$$0 = N + 56 \cdot 1 \cos 31\tfrac{1}{2}^\circ + 35 \sin 31\tfrac{1}{2}^\circ - 4 \times 9 \cdot 5 \sin 31\tfrac{1}{2}^\circ$$
$$- (14 \cdot 8 + 19 \cdot 4 + 23 \cdot 9 + 28 \cdot 1 + 32 + 35 \cdot 5 + 38 \cdot 7) \sin 31\tfrac{1}{2}^\circ.$$

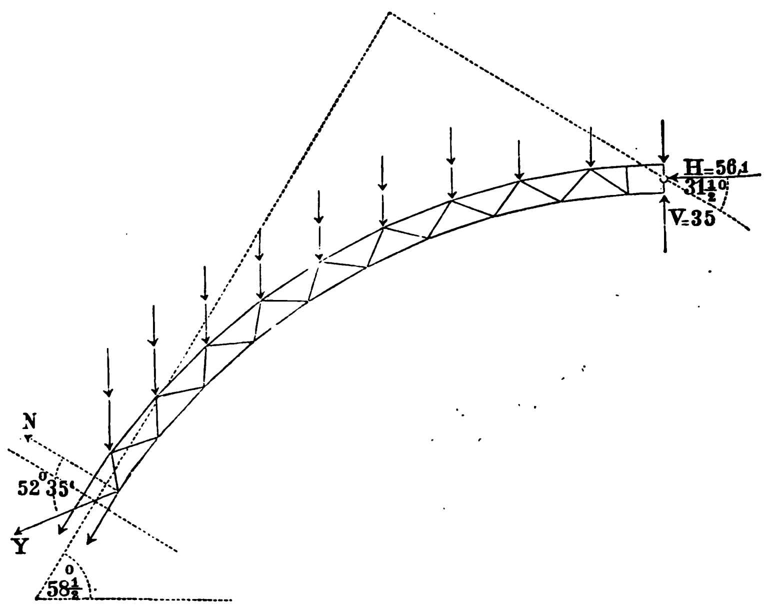
Solving this equation:

$$N \text{ (max.)} = 54 \cdot 3 \text{ tons};$$

and since Y makes an angle of 52° 35′ with N,

$$Y \text{ (max.)} = \frac{54 \cdot 3}{\cos 52^\circ 35'} = + 89 \cdot 3 \text{ tons}.$$

Fig. 257.



To determine Y (min.) the joints 3, 4, 5, 6, 7, 8, 9, are to be unloaded, and in this case the equations to find the hinge-reaction become

$$0 = H \times 50 + V \times 50 - 5595$$
$$0 = - H \times 50 + V \times 50 + 5595 - 506 - 568 - 586 - 562 - 509 - 430 - 337$$
$$H = 77 \qquad V = 35.$$

Hence, Fig. 258, the equation to find N is:

$$0 = N + 77 \cos 31\tfrac{1}{2}^\circ - 35 \sin 31\tfrac{1}{2}^\circ - 4 \times 9{\cdot}5 \sin 31\tfrac{1}{2}^\circ - (5 + 9{\cdot}9) \sin 31\tfrac{1}{2}^\circ;$$

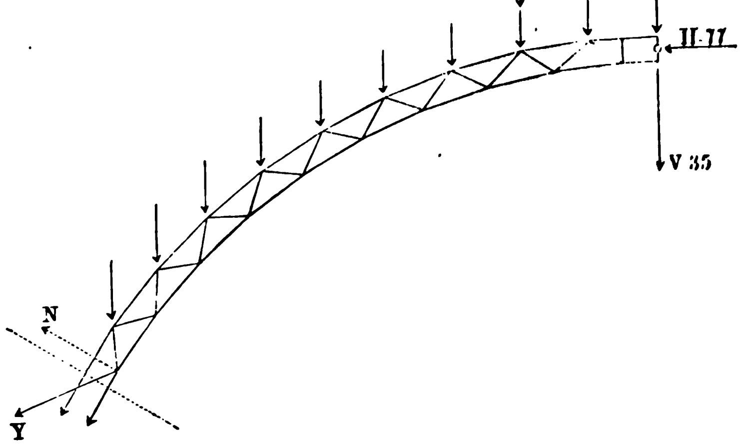
or,

$$N\ (\text{min.}) = -\ 19{\cdot}7\ \text{tons};$$

and the corresponding value of Y is:

$$Y\ (\text{min.}) = \frac{-\ 19{\cdot}7}{\cos 52^\circ 35'} = -\ 32{\cdot}5\ \text{tons}.$$

FIG. 258.



These three examples show sufficiently the mode of calculating the stress in the various bars, and the calculations for the remaining bars will not be given, as they would occupy too much space.

Calculation of the Stress in the Ring.

In a case like the present, where the number of ribs is small, the ring connecting their lower extremities will be a polygon, and from Fig. 259 the following is the equation to find the stress S in the sides of this polygon:

$$2\,S \sin 22\tfrac{1}{2}^\circ = H$$

$$S = \frac{111{\cdot}9}{2 \times 0{\cdot}3827} = 146{\cdot}1\ \text{tons}.$$

If, however, the number of ribs is very large, the horizontal thrust acting on the ring can be considered as uniformly distributed over its total length. Let p be this normal thrust against the inside of the ring per unit of length, then, from Fig. 260, if ϕ is a very small angle:

$$p \,.\, 2\,r\,\phi = 2\,S\,\phi;$$

or,

$$S = p\,r.$$

Fig. 259.

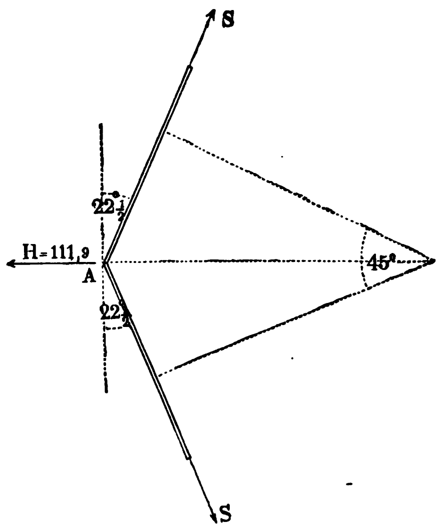
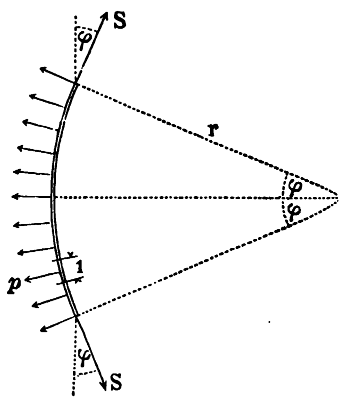


Fig. 260.



In the preceding example:

$$p = \frac{H}{r \,.\, \frac{\pi}{4}} = \frac{111 \cdot 9}{51 \times 0 \cdot 7854} = 2 \cdot 794;$$

and therefore,

$$S = 2 \cdot 794 \times 51 = 142 \cdot 5 \text{ tons.}$$

The difference in this case is so small that it does not matter which method of calculation is adopted.

§ 31.—Generating Curve, for a Dome, requiring the Least Quantity of Material.

In the preceding example it was required to calculate the stresses in a given dome. To simplify the calculations the form was taken as that of a hemisphere, or the generating curve was the quadrant of a circle. If, however, the form of the generating curve necessitating the least quantity of material in the ribs were required, the form of the linear arch (or curve of equilibrium) to carry the unequally distributed loads would have to be found. The principal boom would be made to this curve, a hinge connecting each half would be placed at its vertex. To meet the effect of partial loading a secondary boom, connected to the principal boom by means of diagonals, should be provided. Evidently the secondary boom and the diagonals would have no stress in them when the whole load was on the structure, and also the arithmetical values of the maxima and minima stresses in them would be equal (precisely as in the horizontal and diagonal bars of the braced arch of § 22).

There is no difficulty in finding the required curve if the dome be sufficiently flat, and the number of ribs sufficiently great, for the portions of the surface contained between two adjacent ribs to be considered, without too great an error, as plane triangles, and if the load can be assumed as uniformly distributed over the area of this triangle. In this case the centre of gravity of the triangle S P can be taken as the point of application of the resultant load on the part covered by this triangle (Figs. 261, 262, 263).

Let κ therefore be the load per unit of horizontal surface' and n the number of ribs (consequently $\frac{2\pi}{n}$ the very small angle contained by the horizontal projection of two adjacent ribs), then $x \cdot \frac{2\pi}{n} \cdot \frac{x}{2}$ is the area of the triangle S P, and, taking moments about P (Fig. 263),

$$H\,y = \frac{\kappa\,\pi\,x^2}{n} \cdot \frac{x}{3}. \qquad (1)$$

This equation is also necessarily true for the triangle S A, hence putting

$$x = l \quad \text{and} \quad y = f$$

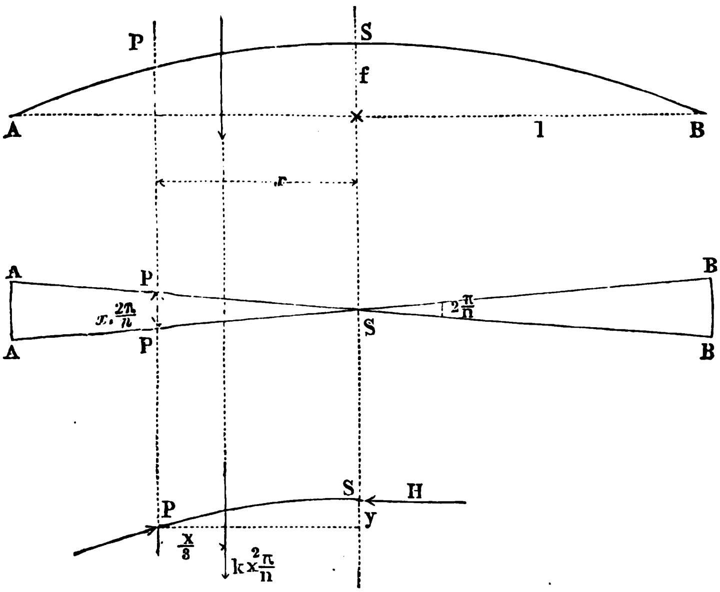
$$\mathrm{H}f = \frac{\kappa \pi l^2}{n} \cdot \frac{l}{3}. \qquad [2]$$

Dividing equation 1 by equation 2,

$$\frac{y}{l} = \frac{x^3}{l^3}.$$

This is the equation to the cubical parabola, and if this form be given to the ribs, the stress in the secondary boom and the diagonals will be zero when the dome is fully loaded.

FIGS. 261, 262, AND 263.



In the above investigation it has been assumed that the weight of the rib itself is very small, or at any rate distributed in the same manner as the other loads. It would, however, be more accurate to consider this load as uniformly distributed along the horizontal projection of the rib, and if p is this

uniform load per unit of length of the span, the above equations become

$$\mathrm{H}\, y = \frac{\kappa \pi x^3}{3\, n} + \frac{p\, x^2}{2} \qquad [1\mathrm{A}]$$

$$\mathrm{H}\, f = \frac{\kappa \pi l^3}{3\, n} + \frac{p\, l^2}{2} \qquad [2\mathrm{A}]$$

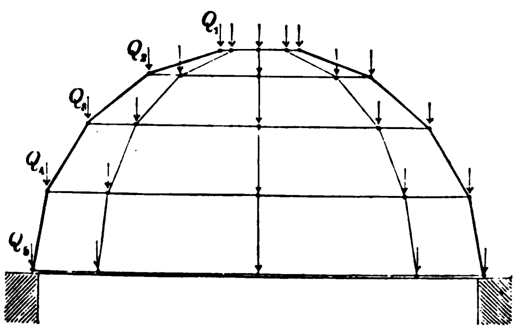
$$\frac{y}{f} = \frac{x^3 + \dfrac{3\, p\, n}{2\, \kappa \pi} \cdot x^2}{l^3 + \dfrac{3\, p\, n}{2\, \kappa \pi} \cdot l^2} \qquad [3\mathrm{A}]$$

If the height of the dome and the number of ribs were such that the above assumption could not be made without sensible error, the required curve would have to be found by following the principles laid down for the determination of linear arches.

§ 32.—Dome Formed of Articulated Ribs and Rings.*

The skeleton of the dome given in Figs. 264 and 265 shows half a regular eighteen-sided polygon in elevation, and a regular octagon on plan. It is assumed that the various bars

Fig. 264.



are connected together by free joints. These joints lie in the surface of a hemisphere of 10 metres radius, and the surface of

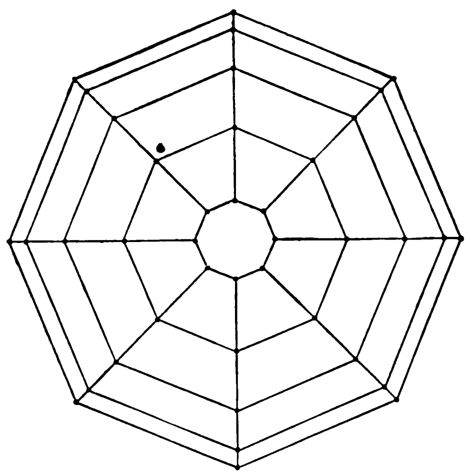
* See 'Berliner Zeitschrift für Bauwesen,' 1866, "The Construction of Domed Roofs," by W. Schwedler.

which is equal to $2\pi r^2 = 2 \times 3{\cdot}1416 \times 10^2 = 628{\cdot}32$ square metres. If therefore the load on the dome be $p = 200$ kilos. per square metre, the sum of the loads on all the joints

$$= G = p \, . \, 2\pi r^2 = 200 \times 628{\cdot}32 = 125664 \text{ kilos.}$$

It can be assumed, without any great error, that the sum of the loads on all the joints in any one of the horizontal rings (Fig. 265) is proportional to the radius of the circle circum-

FIG. 265.



scribing the corresponding octagon (with the exception, however, of the lowest ring, which has only half the load to bear). The radii of these five circles are:

$$\begin{aligned}
r_1 &= r \sin 10^\circ = 10 \times 0{\cdot}17365 = 1{\cdot}7365 \text{ metre} \\
r_2 &= r \sin 30^\circ = 10 \times 0{\cdot}5 \quad\;\;\, = 5{\cdot}0 \quad ,, \\
r_3 &= r \sin 50^\circ = 10 \times 0{\cdot}76604 = 7{\cdot}6604 \quad ,, \\
r_4 &= r \sin 70^\circ = 10 \times 0{\cdot}93969 = 9{\cdot}3969 \quad ,, \\
r_5 &= r \sin 90^\circ = 10 \times 1 \quad\quad\;\; = 10{\cdot} \quad ,,
\end{aligned}$$

Now, x the load on a ring whose radius is unity, can be found from the following equation:

$$1{\cdot}7365\,x + 5\,x + 7{\cdot}6604\,x + 9{\cdot}3969\,x + \frac{10\,x}{2} = 125664;$$

whence:

$$x = 4364 \cdot 3 \text{ kilos.}$$

Thus the loads on the various rings are:

$$\begin{aligned}
x r_1 &= 4364 \cdot 3 \times 1 \cdot 7365 = 7578 \cdot 6 \text{ kilos.*} \\
x r_2 &= 4364 \cdot 3 \times 5 \cdot 0 = 21821 \cdot 5 \quad " \\
x r_3 &= 4364 \cdot 3 \times 7 \cdot 6604 = 33432 \cdot 3 \quad " \\
x r_4 &= 4364 \cdot 3 \times 9 \cdot 3969 = 41010 \cdot 9 \quad " \\
\frac{x r_5}{2} &= \frac{4364 \cdot 3 \times 1}{2} = 21821 \cdot 5 \quad "
\end{aligned}$$

and as each ring contains eight joints, these loads must be divided by 8 to obtain the load on one joint; † thus:

$$\begin{aligned}
Q_1 &= \frac{7578 \cdot 6}{8} = 947 \text{ kilos.} \\
Q_2 &= \frac{21821 \cdot 5}{8} = 2728 \quad " \\
Q_3 &= \frac{33432 \cdot 3}{8} = 4179 \quad " \\
Q_4 &= \frac{41010 \cdot 9}{8} = 5126 \quad " \\
Q_5 &= \frac{21821 \cdot 5}{8} = 2728 \quad "
\end{aligned}$$

where Q_1, Q_2 ... Q_5 denote the loads on the five joints of a rib.

Calculation of the Stresses produced by the Full Load.

Imagine that two sections are taken through the dome by means of vertical planes, as shown in Figs. 266 and 267, and that equilibrium is maintained by forces applied to the end of, and in the direction of, each bar that has been cut through.

The upper ring exerts a horizontal pull R_1 on the rib,

* It is evident that if a lantern or any other load were placed on the top, the load on the top ring would have to be increased accordingly.

† If the number of ribs had been 16 instead of 8, these loads would have to be divided by 16, and so on; otherwise the calculations are the same. A small number of ribs has been chosen to obtain distinct figures.

which is the resultant of the stresses X_1 in the bars of this ring. This is expressed by the equation

$$2\,X_1 \sin \epsilon = R_1 \quad \text{or} \quad X_1 = \frac{R_1}{2 \sin \epsilon};$$

in which

$$\epsilon = 22 \cdot 5^\circ.$$

Fig. 266.

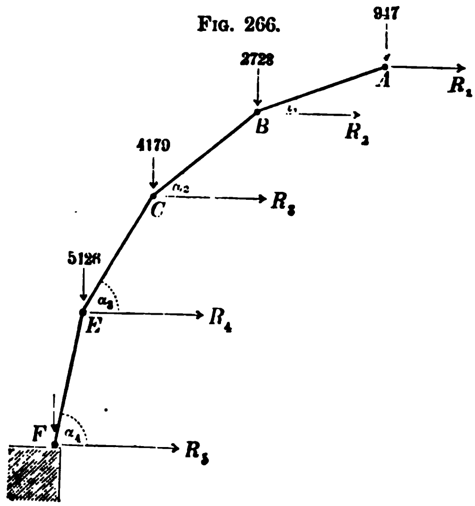
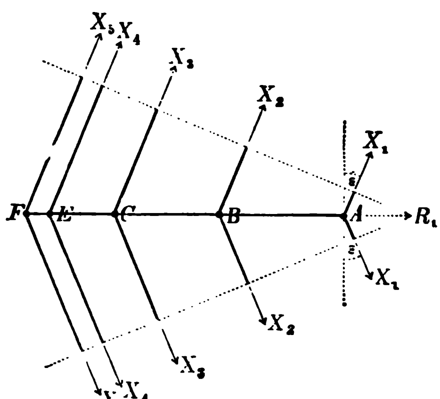


Fig. 267.



The same occurs at each joint, and the stresses in the ring and their resultant are connected by a similar equation.

Further, let the part A B of the rib be cut through, and equilibrium maintained by an applied force D_1 (Fig. 268), and let this force be replaced by its horizontal and vertical components; then, by equating the algebraic sum of the horizontal forces, and also that of the vertical forces, to zero,

$$V_1 = 947 \text{ kilos. and } R_1 = -H_1;$$

now, from the figure,

$$\tan \alpha_1 = \frac{V_1}{H_1}, \qquad \sin \alpha_1 = \frac{V_1}{D_1};$$

and since $\alpha_1 = 20°$, the following values are obtained:

$$H_1 = \frac{947}{\tan \alpha} = +2602 \text{ kilos.}$$

$$D_1 = \frac{947}{\sin 20°} = +2770 \text{ kilos.}$$

$$R_1 = -2602 \text{ kilos.}$$

$$X_1 = -\frac{2602}{2 \sin 22 \cdot 5°} = -3400 \text{ kilos.}$$

FIG. 268.

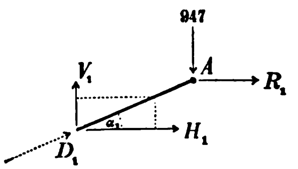
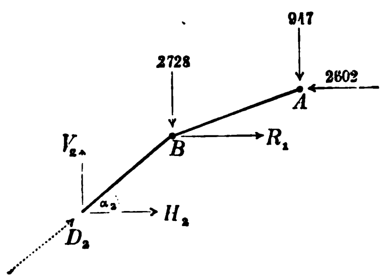


FIG. 269.



The same process applied to the part of the structure shown in Fig. 269, gives:

$$V_2 = 947 + 2728 = +3675 \text{ kilos.}$$

$$H_2 = \frac{V_2}{\tan \alpha_2} = \frac{3675}{\tan 40°} = +4380 \text{ kilos.}$$

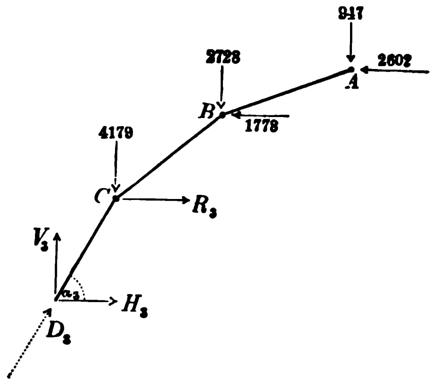
$$D_2 = \frac{V_2}{\sin \alpha_2} = \frac{3675}{\sin 40°} = +5717 \text{ kilos.}$$

$$R_2 = 2602 - H_2 = -1778 \text{ kilos.}$$

$$X_2 = -\frac{1778}{2 \sin 22 \cdot 5°} = -2323 \text{ kilos.}$$

And similarly for Fig. 270:

FIG. 270.

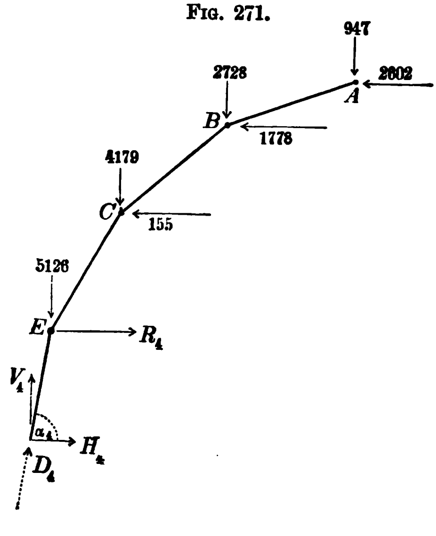


$$V_3 = 947 + 2728 + 4179$$
$$= + 7854 \text{ kilos.}$$
$$H_3 = \frac{V_3}{\tan \alpha_3} = \frac{7854}{\tan 60^\circ}$$
$$= + 4535 \text{ kilos.}$$
$$D_3 = \frac{V_3}{\sin \alpha_3} = \frac{7854}{\sin \alpha_3}$$
$$= + 9069 \text{ kilos.}$$
$$R_3 = 2602 + 1778 - H_3$$
$$= - 155 \text{ kilos.}$$
$$X_3 = \frac{-155}{2 \sin 22 \cdot 5^\circ}$$
$$= - 203 \text{ kilos.}$$

And, lastly, from Fig. 271:

$$V_4 = 947 + 2728 + 4179 + 5126 = + 12980 \text{ kilos.}$$
$$H_4 = \frac{V_4}{\tan \alpha_4} = \frac{12980}{\tan 80^\circ} = + 2289 \text{ kilos.}$$
$$D_4 = \frac{V_4}{\sin \alpha_4} = \frac{12980}{\sin 80^\circ} = + 13180 \text{ kilos.}$$
$$R_4 = 2602 + 1778 + 155 - H_4 = + 2246 \text{ kilos.}$$
$$X_4 = \frac{2246}{2 \sin 22 \cdot 5^\circ} = + 2935 \text{ kilos.}$$

FIG. 271.



V_4, the resolved part vertically of the stress in the bar E F, is produced by the vertical reaction of the point of support; and H_4, the resolved part horizontally, is due to the tension in the bottom ring. Hence

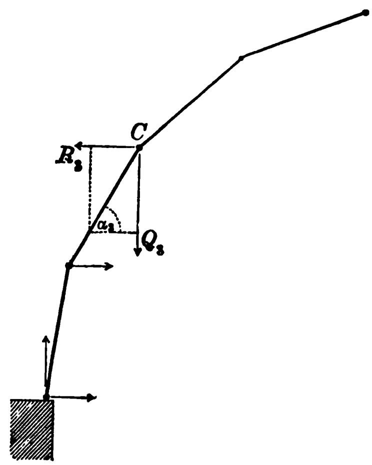
$$R_5 = H_4 = + 2289 \text{ kilos.}$$

$$X_5 = \frac{2289}{2 \sin 22\cdot5^\circ} = + 2991 \text{ kilos.}$$

Minima Stresses in the Rings.

The whole load on the dome will be considered variable; at the same time, however, the loads must always be symmetrical with the vertical axis, or the structure would collapse. If the load on the top ring be removed, it is easy to see that the stress in that ring becomes zero; and likewise that when the load on the second ring is removed, the stress in that ring also becomes zero; and so on for the remaining rings. From this it is evident that the load on any one ring has no influence whatever on the stresses in the rings above it, or, in other words, produces no stress in them; but the load on any ring produces tension in all the rings below it. Consequently, the minimum stress or greatest compression will occur in any ring when it alone is loaded. In Fig. 272 the third ring is represented as loaded, and R_3 is the resultant of the stresses in that ring. For equilibrium, the resultant of R_3 and Q_3 must lie in the direction of the part of rib just below the joint C. Hence

FIG. 272.



$$R_3 \text{ (min.)} = -\frac{Q_3}{\tan \alpha_3} = -\frac{4179}{\tan 60^\circ} = -2413 \text{ kilos.}$$

and similarly for the second and fourth rings:

$$R_2\,(\text{min.}) = -\frac{Q_2}{\tan\alpha_2} = -\frac{2728}{\tan 40^\circ} = -3251 \text{ kilos.}$$

$$R_4\,(\text{min.}) = -\frac{Q_4}{\tan\alpha_4} = -\frac{5126}{\tan 80^\circ} = -904 \text{ kilos.}$$

The minima stresses in the rings are therefore

$$X_2\,(\text{min.}) = -\frac{2251}{2\sin 22{\cdot}5^\circ} = -4248 \text{ kilos.}$$

$$X_3\,(\text{min.}) = -\frac{2413}{2\sin 22{\cdot}5^\circ} = -3153 \text{ kilos.}$$

$$X_4\,(\text{min.}) = -\frac{904}{2\sin 22{\cdot}5^\circ} = -1181 \text{ kilos.}$$

The first and fifth rings are not considered, for in the fifth compression can never occur, and the top ring is always in compression; the value already obtained ($X_1 = -3400$ kilos.) is therefore the minimum stress required.

Maxima Stresses in the Rings.

From what has been already said, it is evident that the maximum stress or greatest tension occurs in any ring when all the rings above it are loaded, itself unloaded, and the rings below it either loaded or unloaded. Thus in Fig. 273 the resultant R_3 of the tensions in the ring at C reaches its maximum value when the two upper rings are loaded, and can be found from the equation (Fig. 274)

$$R_3 = H_2 - H_3 = \frac{V_2}{\tan\alpha_2} - \frac{V_3}{\tan\alpha_3};$$

but

$$V_2 = V_3 = 947 + 2728 = 3675,$$

hence

$$R_3 = \frac{3675}{\tan 40^\circ} - \frac{3675}{\tan 60^\circ} = +2258 \text{ kilos.};$$

and therefore

$$X_3\,(\text{max.}) = +\frac{2258}{2\sin 22{\cdot}5^\circ} = +2950 \text{ kilos.}$$

To find the maximum stress in the second ring, the top ring alone must be loaded, and consequently,

$$V_2 = V_1 = 947,$$

$$\therefore \quad R_2 = \frac{947}{\tan 20^\circ} - \frac{947}{\tan 40^\circ} = +1473,$$

$$X_2\,(\text{max.}) = -\frac{1473}{2 \sin 22 \cdot 5^\circ} = +1925 \text{ kilos.}$$

Fig. 273.

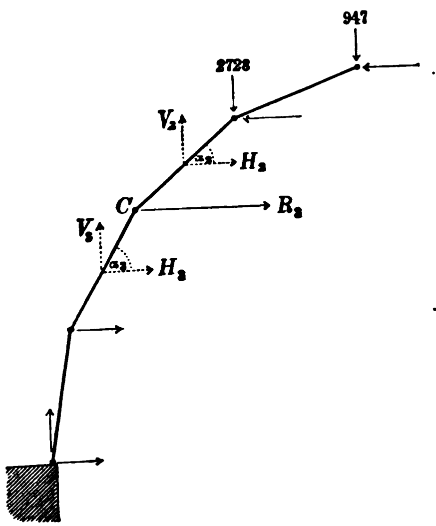
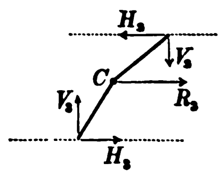


Fig. 274.



Lastly, to determine X_4 (max.) the three upper rings should alone be loaded, and the equations are—

$$V_4 = V_3 = 947 + 2728 + 4179 = 7854,$$

$$R_4 = \frac{7854}{\tan 60^\circ} - \frac{7854}{\tan 80^\circ} = +3150,$$

$$X_4\,(\text{max.}) = +\frac{3150}{2 \sin 22 \cdot 5^\circ} = +4116 \text{ kilos.}$$

In the above calculations it has always been considered that the load on any ring was equally distributed amongst the joints, or, in other words, that the loading was symmetrical

with the axis of the dome. The joints being free, any departure from this symmetrical loading would immediately bring the structure down. To enable the dome to resist unequal loading, either the covering must possess sufficient stiffness to prevent any deformation, or else the free joints must be replaced by fixed joints, and both the ribs and rings strengthened, so that they may prevent deformation by their resistance to bending. The determination of the bending stresses thus called into play cannot, however, be accomplished by elementary means.

TENTH CHAPTER.

§ 33.—Continuous Girder Bridges.

It was seen, in treating of braced arches, that by introducing hinges the stresses in the various bars could be kept between easily controllable limits, and also that the dangers arising from a slight giving way of the abutments, or by changes of temperature, could be totally avoided. But hinges can also be employed with advantage in girder bridges (those that require only vertical reactions at the abutments or piers) when the span is great, and there are two or more openings in succession to be bridged over.

It is found that in such cases a great saving of material is effected by using a continuous girder, instead of several spanning each opening separately. But in these structures, as in braced arches without hinges, there is the danger of a very slight alteration in the position of the supports producing very great differences in the stresses; in braced arches the danger lies in the horizontal displacement of the abutments, but in the present case a vertical displacement becomes critical. Therefore the same reasons that were given with reference to braced arches in § 24 would point to the advisability of breaking the continuity of the girder by means of hinges, and thus making the stresses in the structure independent of small vertical displacements of the points of support. In the case of braced arches, the crown and the abutments were found to be the best places for the hinges; but with girder bridges the best positions are on each side of the central piers, so that the portions of the girder over the piers may act as supports to the other parts (Fig. 275).

The part of the girder resting on either of the piers is to be regarded as supported at two points; and in order that there may be no chance of overturning with a partial load, the dis-

tance of these two points, or the breadth of the pier, must not be less than a certain dimension which will now be found. The worst distribution of the load, as regards the left pier portion, is that shown in Fig. 276, in which only the parts B C and C E are loaded, and the remainder of the bridge unloaded (moving load).

FIG. 275.

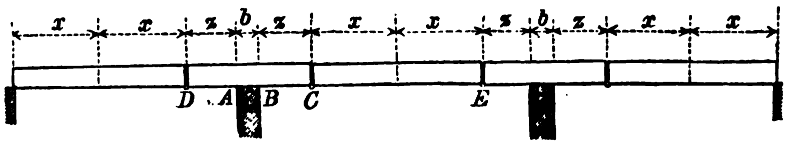
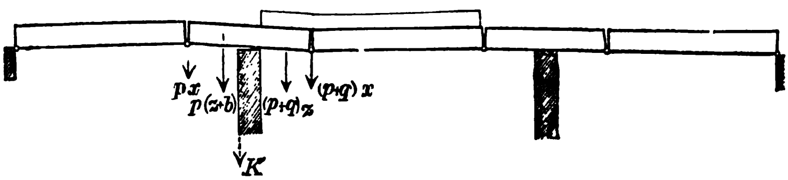


FIG. 276.



The equation of moments about the point B is then

$$0 = (p+q)\,x\,z + (p+q)\,z\,.\,\frac{z}{2} - p\,x\,(z+b) - p\,(z+b)\left(\frac{z+b}{2}\right),$$

where p is the permanent, and q the moving, load per unit of length. Solving this equation, and putting n for the ratio $\frac{q}{p}$,

$$b \geqq -(x+z) + \sqrt{(x+z)^2 + 2\,n\,z\left(x + \frac{z}{2}\right)}.$$

Now, since the ratio n generally increases as the span diminishes, it follows that very small spans would require proportionately very wide piers. To obviate this, the part of the girder over the pier can be anchored down to the masonry by tension rods. With a partial load, a tension, K, is produced in these rods, the moment of which about B ($=$ K b) helps to maintain equilibrium. The equation of moments then becomes

$$0 = (p+q)\,x\,z + (p+q)\,\frac{z^2}{2} - p\,x\,(z+b) - p\,\frac{(z+b)^2}{2} - \mathrm{K}\,b.$$

If this arrangement be adopted, however, the weight Q of the

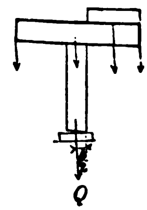
pier must be such as to prevent overturning. This condition is expressed by (Fig. 277)

$$Q\frac{b}{2}+px(z+b)+p\frac{(z+b)^2}{2}\geqq(p+q)z\left(x+\frac{z}{2}\right).$$

If, however, Q becomes greater than is thought advisable, the distance b of the two points A B can be increased by using double piers.

The central and abutment portions being simply supported at the ends can be constructed either as parabolic girders (described in the second chapter), or as braced girders with parallel booms (described in the third chapter). The portions over the piers could also be given this latter form; but a variety of the parabolic form may also be adopted. This variety can be deduced as follows:—

Fig. 277.



When two or more equal and symmetrical chains (either hanging or arch-shaped), having the same load per unit of length of the span, are so placed next each other that the second abutment of the first chain is the first abutment of the second chain, and the second abutment of the second chain is

Fig. 278.

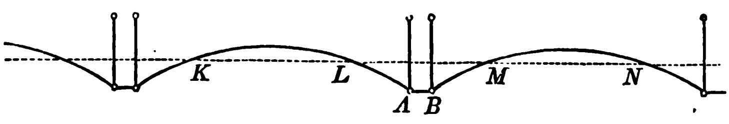
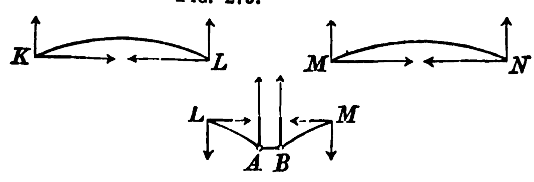


Fig. 279.

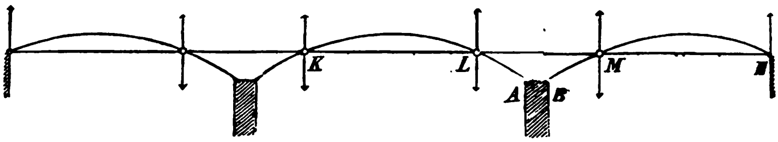


the first abutment of the third chain, and so on, the horizontal tensions or thrusts balance each other at the common abutments, and the reactions are entirely vertical, so that these abutments might be replaced by tension rods. Instead of a single rod, two separated by a horizontal bar, A B, might be used, as already seen in § 27 (Figs. 278 and 279).

If the part L A B M be separated from the two parts K L and M N, the horizontal forces required for equilibrium can be obtained by joining K L, M N, and L M by means of horizontal tie-rods. The vertical forces required for the part L A B M are equal, and opposite to those required for the parts K L and M N, and equilibrium will therefore be maintained if the latter parts be placed on the former in their original position.

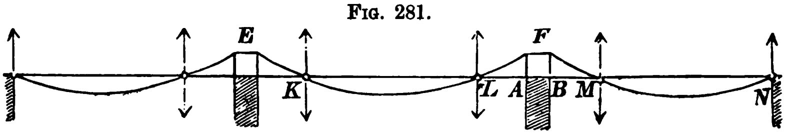
The stresses in the chain have not been altered by the introduction of the horizontal tie-rods, and consequently a bridge constructed as shown in Fig. 280 will, when the load is uni-

FIG. 280.



formly distributed, require no diagonals. Further, since the reasoning for an arch chain applies also to a hanging chain, what has been said above will apply to the structure shown in Fig. 281, the tie-rod becoming a compression bar.

FIG. 281.



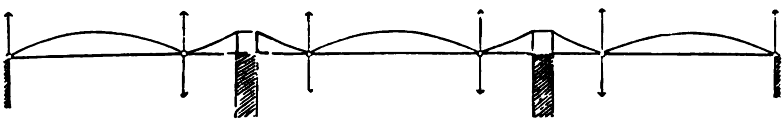
The stresses in the chains can be found from the formula given in § 8. The laws given at page 33 are also applicable namely, that the resolved part vertically of the stress at any point is equal to the load on the bridge between that point and the centre, and that the resolved part horizontally of the stress is constant. It is only an alteration in the height of the arc that changes this horizontal stress.

Evidently the height of arc of the three ordinary parabolic girders in Figs. 280 and 281 can be altered without affecting the portions of the bridge over the piers, for only the vertical reactions are transmitted to these latter; the horizontal stress in the parabolic girders will, however, be altered. The hori-

zontal stress in the pier portions depends solely on the height of arc of the parabolas of which they are formed.

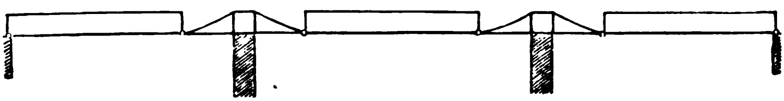
The parabolic girders of Figs. 280 and 281 can be transposed without making any difference in the other parts. Thus

FIG. 282.



a structure of the form shown in Fig. 282 is obtained, which can be adopted with advantage if head room under the bridge is of importance. It is hardly necessary to remark that the

FIG. 283.



parabolic girders can also be replaced by braced girders with parallel booms, and that this will not affect the pier portions (Fig. 283).

§ 34.—CONTINUOUS GIRDER BRIDGE IN THREE SPANS. CENTRAL OPENING, 160 METRES; SIDE OPENINGS, 130 METRES.

The weight of the bridge (Fig. 284) is estimated at 8000 kilos. per metre run, and each girder has half of this to carry. The length of each bay being 10 metres, each joint has 40,000 kilos., or, reckoning 1000 kilos. to the ton, 40 tons dead load to bear. The moving load is taken at 4000 kilos. per metre run, which is equivalent to 20 tons moving load per joint.

The three parabolic girders placed between the piers (Fig. 284) have each a span of 100 metres and a height of 12·5 metres, and the stresses in them can be found as explained in the second chapter (Fig. 39). It will therefore only be

necessary to show how to find the stresses in the parts resting on the piers.

The pressure D produced at C by the girder C E will be greatest when the girder is fully loaded, and least when it is unloaded. In the first case:

$$D = \frac{(40 + 20)\,10}{2} = 300 \text{ tons};$$

and, in the second case:

$$D = \frac{40 \times 10}{2} = 200 \text{ tons.}$$

Fig. 284.

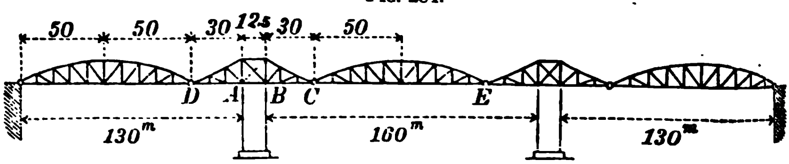
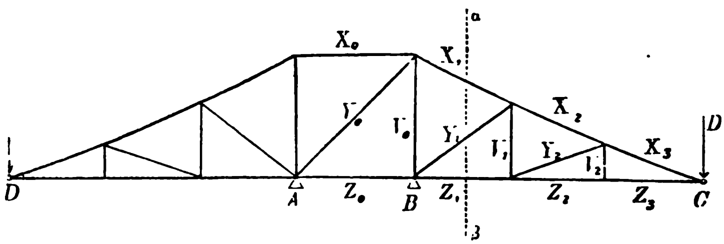


Fig. 285.



Either one or the other of these values will have to be substituted for D, according as it tends to increase or decrease the stress in any bar, and according as the maximum or the minimum stress in that bar is to be determined.

Thus, to find X_1 draw a section line $a\,\beta$ (Figs. 285 and 286) and form the equation of moments with reference to the point B, thus:

$$0 = -X_1 \times 11{\cdot}266 + D \times 30 + 40\,(10 + 20 + \tfrac{30}{2}) + 20\,(10 + 20 + \tfrac{30}{2}).$$

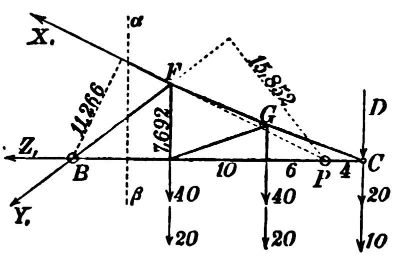
From this equation it appears that D, as well as the loads on the points F, G, C, make X_1 positive. To find X_1 (max.), there-

fore, the girder C E and the joints F, G, C must be loaded, and by substituting the greatest value of D the equation becomes:

$$0 = -X_1 \times 11{\cdot}266 + 300 \times 30 + 60\left(10 + 20 + \tfrac{20}{2}\right)$$
$$X_1\,(\text{max.}) = +\,1038{\cdot}5 \text{ tons.}$$

To determine Y_1 moments must be taken about P (Fig. 286). It is not necessary to consider the permanent load, for, as already remarked in § 33, a uniformly distributed load produces no stress in the diagonals. Further, it is only necessary to calculate either Y_1 (max.) or Y_1 (min.), as both values are numerically equal but of contrary sign, since the moving load when covering the bridge is uniformly distributed. But to show that the method is quite independent of such previous knowledge, the calculations both for Y_1 (max.) and Y_1 (min.) will be made, taking the permanent load in each case into consideration.

Fig. 286.



The equation of moments is:

$$0 = -Y_1 \times 15{\cdot}852 + D \times 4 + 40\left(\tfrac{4}{2} - 6 - 16\right) + 10 \times 4 - 20(6 + 16).$$

To obtain Y_1 (max.) the greatest value of D must be substituted, and the negative terms due to the moving load omitted, thus:

$$0 = -Y_1 \times 15{\cdot}852 + 300 \times 4 + 40\left(\tfrac{4}{2} - 6 - 16\right) + 10 \times 4\,;$$

or,

$$Y_1\,(\text{max.}) = +\,27{\cdot}76 \text{ tons.}$$

And to find Y_1 (min.), D must be given its least value and the positive terms due to the moving load must be left out, thus:

$$0 = -Y_1 \times 15{\cdot}852 + 200 \times 4 + 40\left(\tfrac{4}{2} - 6 - 16\right) - 20(6 + 16)$$
$$Y_1\,(\text{min.}) = -\,27{\cdot}76 \text{ tons.}$$

The stress in the bar Z_1 can be found from the following equation of moments, formed with reference to the point F:

$$0 = Z_1 \times 7{\cdot}692 + D \times 20 + 40\left(\tfrac{20}{2} + 10\right) + 20\left(\tfrac{20}{2} + 10\right);$$

from this equation it appears that all the loads produce compression, therefore putting $D = 300$

$$0 = Z_1 \times 7\cdot 692 + 300 \times 20 + 40\left(\tfrac{20}{2} + 10\right) + 20\left(\tfrac{20}{2} + 10\right);$$

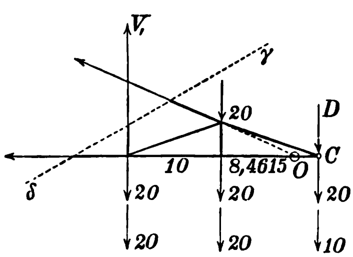
or,

$$Z_1 \text{ (min.)} = -936 \text{ tons.}$$

V_1 is found by taking moments about O (Fig. 287). Half the dead load will be considered as applied to the lower joints, and the other half to the upper joints; the equation of moments is:

$$0 = V_1 \times 18\cdot 4615 + D \times 1\cdot 5385 + 40\left(\tfrac{1\cdot 5385}{2} - 8\cdot 4615 - \tfrac{18\cdot 4615}{2}\right)$$
$$+ 20 \times \tfrac{1\cdot 5385}{2} - 20(8\cdot 4615 + 18\cdot 4615);$$

Fig. 287.



V_1 is therefore greatest when D is least, and the joint C unloaded. Consequently:

$$0 = V_1 \times 18\cdot 4615 + 200 \times 1\cdot 5385$$
$$+ 40\left(\tfrac{1\cdot 5385}{2} - 8\cdot 4615 - \tfrac{18\cdot 4615}{2}\right)$$
$$- 20(8\cdot 4615 + 18\cdot 4615)$$
$$V_1 \text{ (max.)} = +49\cdot 2 \text{ tons;}$$

and V_1 is least when D is greatest and the joint C alone is loaded, hence:

$$0 = V_1 \times 18\cdot 4615 + 300 \times 1\cdot 5385 + 40\left(\tfrac{1\cdot 5385}{2} - 8\cdot 4615 - \tfrac{18\cdot 4615}{2}\right)$$
$$+ 20 \times \tfrac{1\cdot 5385}{2}.$$
$$V_1 \text{ (min.)} = +10\cdot 8 \text{ tons.}$$

(The stresses produced by the moving load alone are:

$$+29\cdot 2 \text{ and } -9\cdot 2,$$

which, added to the stress due to the dead load alone, or $+20$, gives the values found above).

For the remaining bars the following equations and results are obtained:

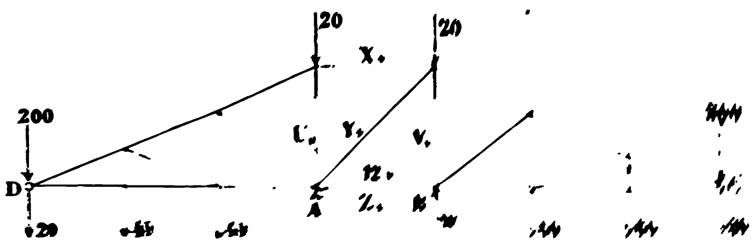
$$0 = -X_2 \times 7\cdot 1 + 300 \times 20 + (40 + 20)\left(\tfrac{20}{2} + 10\right)$$
$$X_2 \text{ (max.)} = +1014 \text{ tons}$$
$$0 = -Y_2 \times 6\cdot 138 + (200 + 100)\, 1\cdot 5385 + 40\left(\tfrac{1\cdot 5385}{2} - 8\cdot 4615\right)$$
$$+ 20 \times \tfrac{1\cdot 5385}{2} - 20 \times 8\cdot 4615$$
$$Y_2 \begin{Bmatrix}\text{max.}\\ \text{min.}\end{Bmatrix} = \pm 27\cdot 57 \text{ tons}$$

$$0 = Z_2 \times 3\cdot 5[illegible] - 300 \times 10 - 30 \times 10$$
$$Z_2\,(\text{min.}) = - [illegible] \text{ tons}$$
$$0 = Y_2 \times 10 - 30 \times 10 - 30 \times 10$$
$$Y_2\,(\text{max.}) = + 60 \text{ tons} \qquad Y_2\,(\text{min.}) = [illegible]$$
$$0 = - X_3 \times 3\cdot 525 + 300 \times 10 + 30 \times 10$$
$$X_3\,(\text{max.}) = + [illegible] \text{ tons}$$
$$0 = Z_3 \times 3\cdot 526 + 300 \times 10 + 30 \times 10$$
$$Z_3\,(\text{min.}) = - 936 \text{ tons}$$

The calculations to find the stresses in the diagonal, and in the two verticals immediately above the pier, vary slightly from the above, on account of the reaction of the pier. This reaction W (Fig. 288) is the pressure the fixed point B exerts against the structure.

To determine W, C D can be considered as a lever having its fulcrum at the other point of support A. It is evident that

FIG. 288.



when W is to be as great as possible, [illegible] of A must be loaded, [illegible] the load on the joint [illegible] no influence on the magnitude of W [illegible] equation of moments is

$$[illegible]$$

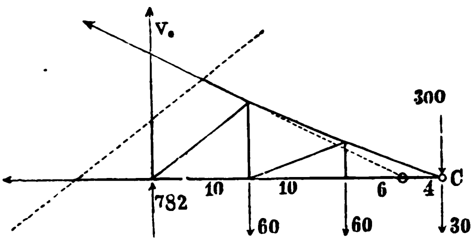
[illegible]

The equation of moments for V_0 (min.) is therefore:

$$0 = V_0 \times 26 + 782 \times 26 + 300 \times 4 + 30 \times 4 - 60(6 + 16)$$
$$V_0 \text{ (min.)} = -782 \text{ tons.}$$

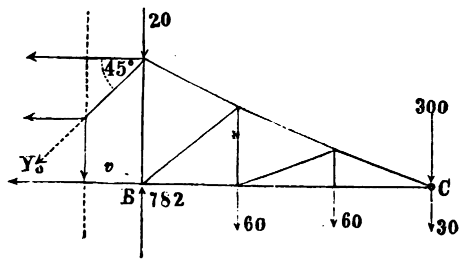
The same loading makes Y_0 a maximum. For since X_0 and Z_0 are parallel, the equation of moments reduces to the

FIG. 289.



condition that the sum of the vertical forces should vanish. Hence v, the resolved part vertically of Y_0 (max.), is equal to the difference between W and the loads on the part of the structure

FIG. 290.



shown in Fig. 290, and this difference is greatest when this part is fully loaded; for the reaction produced at B by any such load is greater than the load itself. Consequently

$$0 = v - 782 + 20 + 60 + 60 + 30 + 300$$
$$v = 312 \text{ tons};$$

and since the diagonal makes an angle of 45° with the vertical:

$$Y_0 \text{ (max.)} = \frac{v}{\cos 45^\circ} = 312\sqrt{2} = +441 \cdot 2 \text{ tons.}$$

As already observed, $Y_0 = 0$ when the bridge is fully loaded, hence the numerical value of Y_0 (min.) must be equal to that of Y_0 (max.). Therefore:

$$Y_0 \begin{Bmatrix} \text{max.} \\ \text{min.} \end{Bmatrix} = \pm 441 \cdot 2 \text{ tons.}$$

As to the stress in the vertical U_0, it is evident that the only vertical force acting upon it at the top, besides the permanent load of 20 tons, is the vertical component of the stress in the top boom, and this component is evidently greatest with a full load. Consequently:

$$- U_0 = 20 + (60 + 60 + 30 + 300)$$
$$U_0 \text{ (min.)} = - 470 \text{ tons.}$$

(The maximum stresses in both the verticals over the pier are negative, and it is therefore not necessary to consider them. By reversing, however, the loading in Fig. 288, it is easily found that U_0 (max.) $= - 320$ tons.)

The stresses in X_0 and Z_0 are, as in the case of the other horizontal bars, equal to the horizontal stress in the fundamental parabolic chain. Consequently:

$$X_0 \text{ (max.)} = + 936 \text{ tons}$$
$$Z_0 \text{ (min.)} = - 936 \text{ tons.}$$

The value of this horizontal stress depends, as already remarked, on the height of the arc of the parabola E K L F, Fig. 281, to which the part C D belongs. In the present case the vertex of the parabola is 8·0128 metres below the horizontal K L (Fig. 281), and the height of arc is therefore:

$$f = 12 \cdot 5 + 8 \cdot 0128 = 20 \cdot 5128 \text{ metres.}$$

The corresponding span is:

$$2\,l = 160 \text{ metres.}$$

The stress in the horizontal bars can therefore be obtained from the formula (see p. 32):

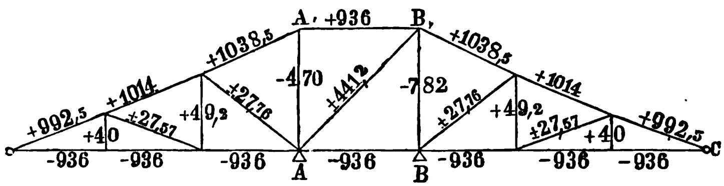
$$H = \frac{(p + q)\,l^2}{2\,f} = \frac{(4 + 2)\,80^2}{2 \times 20 \cdot 5128} = 936 \text{ tons.}$$

(The stress in the horizontal bars of the central parabolic girder is only:

$$\frac{(4+2) \times 50^2}{2 \times 12 \cdot 5} = 600 \text{ tons.})$$

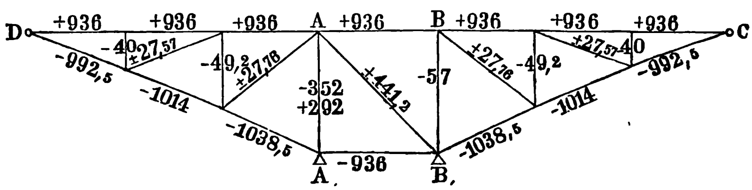
The results obtained by the above calculations are collected together in Fig. 291. By changing the signs of the stresses given in this figure, those for a similar girder turned upside down are obtained. The points A and B will, however, still be

FIG. 291.



the points of support. But in this case A_1, B_1 will generally be chosen as points of support, and the stresses in the two verticals over the pier will consequently be altered. These new stresses can either be determined directly, or else by employing secondary verticals, a method which has been used

FIG. 292.



before (see § 12). The stresses thus obtained are given in Fig. 292.

From Figs. 291 and 292 several derived forms can be obtained, as was done in previous cases (see §§ 7, 11, and 16), but only the alterations that can be made in the construction of the central bay over the pier will be considered.

If, for instance, there are two diagonals in the central bay, both of which are capable of resisting either tension or compression (Fig. 293), the stresses in each diagonal will be exactly

one-half of the stress found above for the diagonal, and the stresses in the two verticals will each be the arithmetical mean of those already obtained (it is easy to satisfy oneself of this by imagining two such girders, with halved stresses, placed one behind the other, one with the central diagonal inclined to the right, and the other with this diagonal inclined to the left).

FIG. 293.

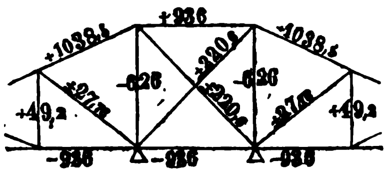


FIG. 294.

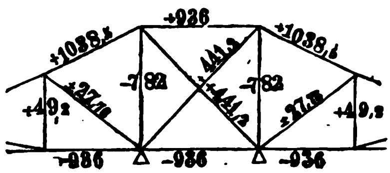


FIG. 295.

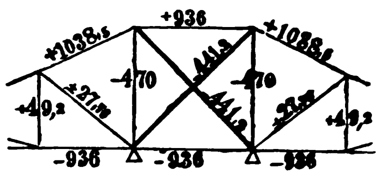


FIG. 296.

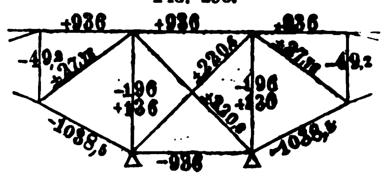


FIG. 297.

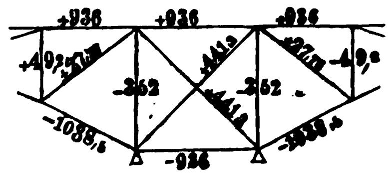
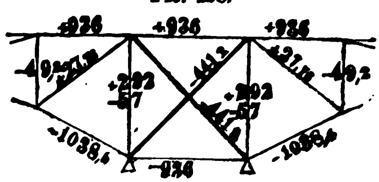


FIG. 298.



If, however, both diagonals can only take up tension or can only resist compression, the stresses given in Figs. 294 and 295 respectively will be those required. The inability of the diagonals to resist tension is expressed in Fig. 295 by double lines.

In a similar manner the structures shown in Figs. 296, 297, and 298 can be derived from Fig. 292.

In all the preceding calculations the load on the joints over the piers has, for simplicity, been taken the same as on the other joints, though accurately speaking the load on these joints is slightly greater on account of the pier being a little wider than a bay.

This width is necessary to prevent the pier girders overturning when subject to the action of a partial load. By substituting in the formula of § 33, viz.

$$b \geqq -(x+z)+\sqrt{(x+z)^2+2nz\left(x+\frac{z}{2}\right)},$$

the values

$$x = 50, \qquad z = 30, \qquad n = \tfrac{20}{40},$$

it is found that,

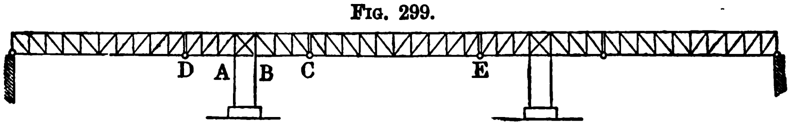
$$b \geqq 11 \cdot 38 \text{ metres.}$$

The width of the pier assumed above, namely 12·5 metres, is therefore a little in excess (all the more so as the permanent load of the central girders is a little less than that of the pier girders, although in the calculations it has been taken as equal).

Continuous Girder with Parallel Booms.

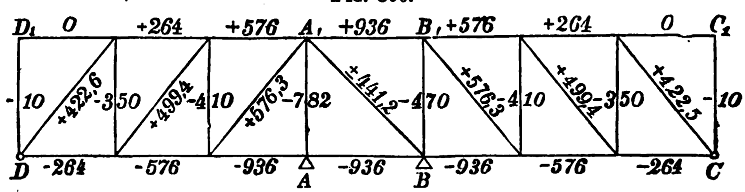
The whole of the continuous girder may be constructed with parallel booms as shown in Fig. 299, or else only parts resting on the piers may be so designed. The stresses in

FIG. 299.



these latter can be obtained by an exactly similar process to that followed above, and these stresses are given in Figs. 300, 301, 302, 303, and 304. To form some idea of the relative quantities of material required by the two designs, the span and height of the girders are the same as in the former

FIG. 300.



example. It must also be observed that all the figures refer to girders carrying the line of railway on their lower booms, and also that the hinges connect the lower booms. Further, in

FIG. 301.

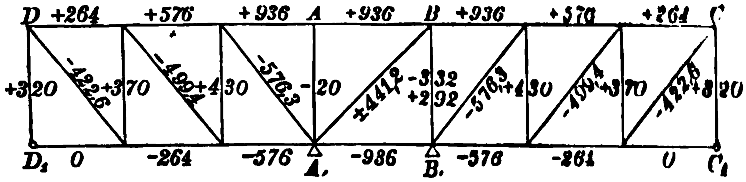


FIG. 302.

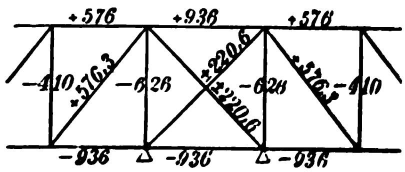


FIG. 303.

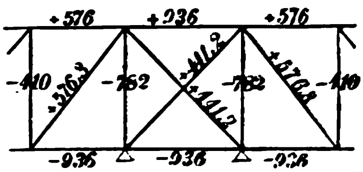


FIG. 304.

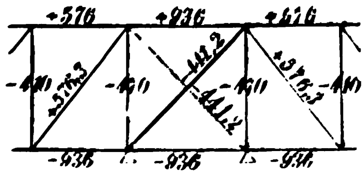


Fig. 302 both the diagonals of the central bay are capable of resisting either tension or compression; in Fig. 303 they can only resist tension, and in Fig. 304, compression.

§ 35.—TO DETERMINE THE SUBDIVISION OF THE WHOLE SPAN REQUIRING THE LEAST QUANTITY OF MATERIAL.

By comparing the stresses given in Fig. 291 it appears, first, that the stresses in the diagonals and verticals are small compared with those in the booms; and secondly, that the stresses in the curved part of the bow do not differ materially from each other or from the stress in the horizontal boom. Now, as the quantity of

material can be taken as nearly proportional to the stress, it follows that by far the largest quantity is contained in the booms, and that it is nearly equally distributed between them. These remarks also apply to the parabolic central girder, as will at once appear by reference to § 6.

Therefore it cannot be far from the truth to assert that the quantity of material in the bridge is proportional to that contained in the horizontal boom.

The above problem resolves itself therefore into the following: To find what subdivision of the span gives the least quantity of material in the horizontal boom.

To solve this problem it is first necessary to find the most advantageous position of the hinges in the central span, and also the most advantageous position of the hinges in both the side spans.

a. Subdivision of the Central Span.

Let the parts C E (Fig. 306) and C A (Fig. 307) be cut out of Fig. 305 and equilibrium maintained by applying the forces H and H_1 respectively, which are the stresses in the booms. Taking moments about S for C E and about A_1 for C A.

$$H \cdot h = p\,x \cdot x - p\,x \cdot \frac{x}{2} \qquad [1]$$

$$H_1 \cdot h = p\,x\,(l - x) + p\,(l - x)\left(\frac{l - x}{2}\right) \qquad [2]$$

Now the sectional area of the booms can be found by dividing the stress in them by S, the safe stress of the material per unit of area. Therefore if F is the sectional area of the boom C E, and F_1 that of C A,

$$F = \frac{H}{S}, \quad F_1 = \frac{H_1}{S};$$

and by substituting the values of H and H_1 from equations 1 and 2,

$$F = \frac{p\,x^2}{2\,h\,S} \qquad [3]$$

$$F_1 = \frac{p\,(l^2 - x^2)}{2\,h\,S}. \qquad [4]$$

By multiplying these sectional areas by the length of the corresponding boom the quantity of material in C E and C A respectively is obtained, thus,

$$M = F x = \frac{p x^2}{2 h S} \qquad [5]$$

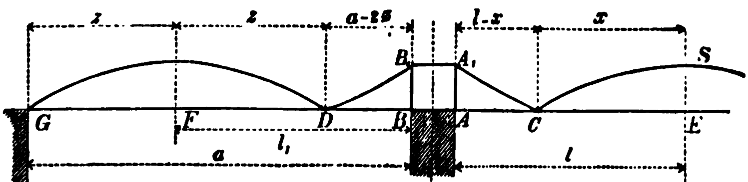
$$M_1 = F_1 (l - x) = \frac{p (l^2 - x^2)(l - x)}{2 h S}; \qquad [6]$$

and the amount of material in A E is

$$(M + M_1) = \frac{p}{2 h S}(l^3 - l^2 x - l x^2 + 2 x^3), \qquad [7]$$

and $(M + M_1)$ is to be a minimum.

FIG. 305.



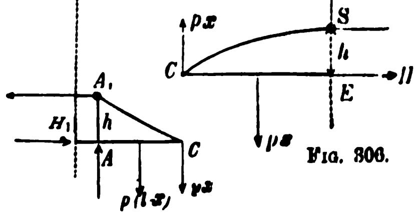


FIG. 307.

FIG. 306.

If in this equation x really represents the value that makes $(M + M_1)$ a minimum, it is evident that the addition of a very small quantity $\pm \Delta$ to x (or, in other words, replacing x by $[x \pm \Delta]$) must have the effect of increasing $(M + M_1)$. But this can only be the case if the first of the three terms added to the expression in bracket by changing x to $x \pm \Delta$, namely,

$$\pm \Delta(-l^2 - 2 l x + 6 x^2) + \Delta^2 (6 x - l) \pm 2 \Delta^3,$$

is equal to zero. For otherwise, by judiciously choosing Δ the first term (which would then be large in comparison to the other two) could be made negative; and consequently $(M + M_1)$

would be diminished. Therefore the condition for a minimum is that

$$-l^2 - 2lx + 6x^2 = 0;^* \qquad [8]$$

whence the following values for $\frac{x}{l}$ and $\frac{l-x}{l}$ are obtained:

$$\frac{x}{l} = \frac{1 + \sqrt{7}}{6} = 0{\cdot}6076 \qquad [9]$$

$$\frac{l-x}{l} = 0{\cdot}3924; \qquad [10]$$

or the lengths A C and C E must be approximately in the ratio 0·4 : 0·6, or 2 : 3.

By substituting the value of x from equation 8 in equation 7, the quantity of material in the horizontal boom from A to E is obtained:

$$M + M_1 = 0{\cdot}47184 \frac{p\,l^3}{2\,S\,h}. \qquad [11]$$

b. Subdivision of the Side Spans.

The quantity of material J in the horizontal boom D F (Fig. 305) can be found by substituting z for x in equation 5, thus:

$$J = \frac{p\,z^3}{2\,h\,S}. \qquad [12]$$

And J_1, the amount of material in B D, can be obtained from equation 6 by writing l_1 for l and z for x.

$$J_1 = \frac{p\,(l_1^2 - z^2)\,(l_1 - z)}{2\,h\,S};$$

or replacing l_1 by $a - z$

$$J_1 = \frac{p\,a\,(a - 2\,z)^2}{2\,h\,S}. \qquad [13]$$

The quantity of material in the horizontal boom from B to G is therefore

$$2\,J + J_1 = \frac{p}{2\,h\,S}(a^3 - 4\,a^2\,z + 4\,a\,z^2 + 2\,z^3). \qquad [14]$$

---

* Or, in other words, the first differential coefficient of the expression in brackets of equation 7 must be equated to zero.

If $z \pm \Delta$ is written for z, the terms in brackets are increased by the expression

$$\pm \Delta(-4a^2 + 8az + 6z^2) + \Delta^2(4a + 6z) \pm 2\Delta^3;$$

and as before the value of z that makes $2J + J_1$ a minimum can be found from the equation

$$-4a^2 + 8az + 6z^2 = 0; \qquad [15]$$

whence the following values for z and $a - 2z$ are obtained:

$$z = \tfrac{1}{3}a(-1 + \sqrt{2 \cdot 5}) = 0 \cdot 3874\, a \qquad [16]$$

$$a - 2z = 0 \cdot 2252\, a; \qquad [17]$$

from which

$$\frac{z}{a - z} = \frac{z}{l_1} = 0 \cdot 6324 \qquad [18]$$

is obtained.

By substituting in equation 14 the value found for z in equation 16, the least quantity of material in the boom from B to G is found to be

$$2J + J_1 = 0 \cdot 16706 \frac{pa^3}{2hS}. \qquad [19]$$

c. *Proportion of the Central Span to the Side Spans.*

The preceding numerical values for $\frac{x}{l}$ and $\frac{z}{l_1}$ are quite independent of the span of their respective openings, and therefore also of the ratio

$$\frac{a}{2l} = n;$$

according to which the whole span is divided into three spans,

$$a, \qquad 2l, \qquad a.$$

The converse, however, is not true, and in fact the most advantageous division of the whole span depends on the subdivision of the single spans.

It will be considered that the single spans have been subdivided in the most economical manner in accordance with equations 9 and 18.

In this case the quantity of material required for half the whole span L is found by adding equations 11 and 19 together, thus:

$$M + M_1 + 2\,J + J_1 = \frac{p}{2\,h\,S}\,(0{\cdot}47184\,l^3 + 0{\cdot}16706\,a^3);$$

or writing L − a for l,

$$M + M_1 + 2\,J + J_1 = \frac{p}{2\,h\,S}\,(0{\cdot}47184\,(L - a^3) + 0{\cdot}16706\,a^3). \qquad [20]$$

As before by changing a into $a \pm \Delta$ it will be found that if the quantity of material is to be a minimum the condition

$$-\,3 \times 0{\cdot}47184\,(L - a)^2 + 3 \times 0{\cdot}16706\,a^2 = 0 \qquad [21]$$

must obtain. Whence

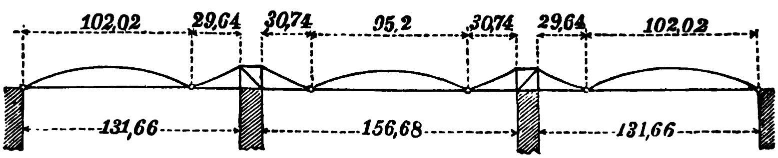
$$\frac{a}{L - a} = \sqrt{\frac{0{\cdot}47184}{0{\cdot}16706}} = 1{\cdot}6806;$$

or writing l instead of (L − a) and n instead of $\frac{a}{2\,l}$

$$n = 0{\cdot}8403. \qquad [22]$$

Applying the results obtained in equations 9, 18, and 22 to the previous numerical example, it will be found that the whole span, 420 metres, is subdivided as shown in Fig. 308.

Fig. 308.



It will be observed that the span of the central parabolic girder is rather smaller than that of the two side ones. This would slightly increase the expense of execution, and owing also to the unsymmetrical arrangement and to the unequal loading of the hinges, the diagonals in the bays immediately above the central piers would be in a state of stress even with a distributed load, thereby slightly adding to the quantity of material.

When it is considered also that the calculations in the above investigation are only approximate, it seems better to choose the simpler ratios

$$n = \frac{130}{160}, \qquad \frac{x}{l} = \frac{50}{80}, \qquad \frac{x}{l_1} = \frac{50}{80},$$

making the girders symmetrical with respect to the central piers.

Equations 1 and 2 also give the greatest stress in the booms of a girder with parallel booms. It is true that in such girders the stress in the booms decreases from the centre to the points of support, but this is more or less compensated by the dimensions of the diagonals and verticals increasing towards the abutments.

It can therefore be assumed that the quantity of metal in such girders is nearly the same as that in parabolic girders—an assumption which will be found justified by comparing Fig. 27 with Fig. 57, and Fig. 291 with Fig. 300. The premises are therefore approximately the same as in the case of parabolic girders, and consequently equations 9, 18, and 22 may be considered as approximately true for girders with parallel booms.

ELEVENTH CHAPTER.

§ 36.—Determination of the Turning Points and Lever Arms by Calculation.

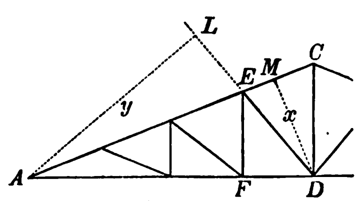
Although a drawing to scale of the structure under consideration can generally be made, upon which the turning points can be found by construction, and the lever arms, with ample accuracy, by measurement, yet cases may occur where it is necessary to obtain these data by calculation and without the help of a drawing. In the following it will be shown that this is by no means difficult, and that when the structure is composed of straight bars the required results can be obtained simply by the comparison of two similar triangles. The examples have been chosen from the various structures already considered, and will therefore render the former calculations more complete.

Roof of § 3.

To find the lever arm x of the stress X in Fig, 10, the similarity of the two triangles D M C and A D C can be employed thus (Fig. 309):

$$\frac{x}{DC} = \frac{AD}{AC},$$

Fig. 309.



and putting

$$CD = 20, \qquad AD = 50,$$
$$AC = \sqrt{50^2 + 20^2},$$

as given in Fig. 8.

$$x = \frac{20 \times 50}{\sqrt{50^2 + 20^2}} = 18 \cdot 6.$$

The lever arm y of the stress Y can be ascertained by comparing the two similar right-angled triangles A L D and E F D, obtaining the equation

$$\frac{y}{AD} = \frac{EF}{ED};$$

or substituting

$$A D = 50, \quad E F = 15, \quad E D = \sqrt{12 \cdot 5^2 + 15^2},$$

$$\therefore y = \frac{50 \times 15}{\sqrt{12 \cdot 5^2 + 15^2}} = 38 \cdot 4.$$

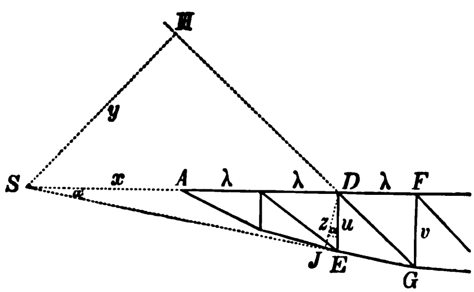
Parabolic Girder (§ 6).

To ascertain the stresses V_2 and Y_3 the position of the turning point S (Fig. 26) had to be determined. This can be done from the equation

$$\frac{u}{x + 2\lambda} = \frac{v}{x + 3\lambda} = \tan \alpha,$$

obtained from Fig. 310 by comparing the similar right-angled triangles S D E and S F G.

FIG. 310.



Now, according to Fig. 21,

$$\lambda = 2 \text{ metres}, \quad u = 1 \cdot 5 \text{ metre}, \quad v = 1 \cdot 875 \text{ metre}.$$

Therefore

$$x = \left(\frac{3u - 2v}{v - u}\right)\lambda = 4 \text{ metres}.$$

Again, from the similarity of the two right-angled triangles S D H and G D F, the following equation is obtained:

$$\frac{y}{S D} = \frac{v}{G D};$$

from which, by substituting

$$S\,D = x + 2\lambda = 8 \text{ metres},$$

and

$$G\,D = \sqrt{v^2 + \lambda^2} = \sqrt{1 \cdot 875^2 + 2^2}.$$

$$y = \frac{8 \times 1 \cdot 875}{\sqrt{1 \cdot 875^2 + 2^2}} = 5 \cdot 47 \text{ metres.}$$

Lastly, to find the lever arm $D\,J = z$ (for the moment of the stress Z_3 about D), the following equation is deduced from Fig. 310,

$$z = u \cos \alpha = u \frac{S\,D}{S\,E} = 1 \cdot 5 \frac{8}{\sqrt{8^2 + 1 \cdot 5^2}} = 1 \cdot 474 \text{ metre.}$$

Sickle-shaped Truss (§ 15).

The equation for x found above can also be put in the form

$$x + 2\lambda = \frac{\lambda}{\frac{v}{u} - 1}.$$

This equation can be adapted to another parabola whose ordinates are n times those of the former, by writing $n\,v$ for v and $n\,u$ for u. The ratio $\frac{v}{u}$ in the denominator of the above equation becomes $\frac{n\,v}{n\,u}$, or remains unchanged. Consequently the intersection of the chord E G, with the horizontal through the points of support is independent of the height of the arc of the parabola.

From this it follows that in the sickle-shaped truss of Fig. 311 the intersection of the chords H N and M J lies in the horizontal through the points of support. The position of O can therefore be found, as in the preceding case, from the equation

$$\frac{u}{x + 3\lambda} = \frac{v}{x + 4\lambda}.$$

Or substituting from Fig. 114 the values

$$\lambda = 1, \quad u = 0 \cdot 710, \quad v = 0 \cdot 852,$$

$$x = \frac{0 \cdot 710}{0 \cdot 852 - 0 \cdot 710} - 3 = 2.$$

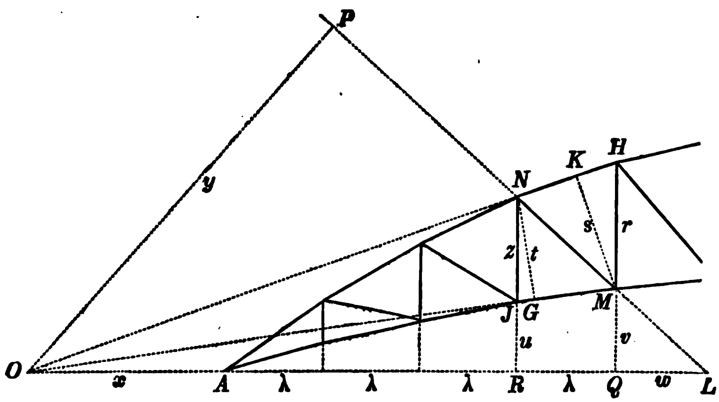
The position of the point L can be determined by means of the similar triangles N R L and M Q L, thus:

$$\frac{w}{v} = \frac{w + \lambda}{u + z}, \text{ or } \frac{w}{v} = \frac{\lambda}{u + z - v};$$

but from Fig. 114, $z = 1 \cdot 065$, therefore

$$w = \frac{0 \cdot 852}{1 \cdot 775 - 0 \cdot 852} = 0 \cdot 9231.$$

FIG. 311.



To obtain the lever arm y (for the stress Y_4 in Fig. 116) the similarity of the two right-angled triangles O P L and M Q L gives the equation

$$\frac{y}{\text{O L}} = \frac{v}{\text{M L}},$$

in which

$$\text{O L} = 2 + 4 + 0 \cdot 9231 = 6 \cdot 9231,$$

and

$$\text{M L} = \sqrt{0 \cdot 852^2 + 0 \cdot 9231^2} = 1 \cdot 256.$$

The value of y thus found is

$$y = \frac{6 \cdot 9231 \times 0 \cdot 852}{1 \cdot 256} = 4 \cdot 7.$$

The position of O being known, the lever arm t (for the stress Z_4) can be found by comparing the similar right-angled triangles J G N and J R O, thus:

$$\frac{t}{z} = \frac{\text{O R}}{\text{O J}};$$

and since $OR = 5$, and $OJ = \sqrt{5^2 + 0 \cdot 71^2}$,

$$t = \frac{5 \times 1 \cdot 065}{\sqrt{5^2 + 0 \cdot 71^2}} = 1 \cdot 054.$$

If, however, the position of O were not known it would be better to find t from the equation

$$\frac{t}{s} = \frac{\lambda}{JM}.$$

The lever arm s (for the stress X_4) can be obtained from the equation

$$\frac{s}{r} = \frac{OQ}{OH},$$

in which

$$r = 1 \cdot 278,\ OQ = 6, \text{ and } OH = \sqrt{6^2 + 2 \cdot 13^2},$$

whence

$$s = \frac{6 \times 1 \cdot 278}{\sqrt{6^2 + 2 \cdot 13^2}} = 1 \cdot 205.$$

But if the position of O were not known this lever arm could be more easily obtained from the equation

$$\frac{s}{r} = \frac{\lambda}{NH}.$$

To calculate the stress V_3, it is necessary to know the point of intersection of the two parabolic chords NF and MJ. From Fig. 312,

$$NT = \rho \tan \alpha = z + \rho \tan \epsilon,$$

or

$$\rho = \frac{z}{\tan \alpha - \tan \epsilon}.$$

Here

$$z = 1 \cdot 065 \quad \text{and} \quad \tan \epsilon = \frac{MQ - JR}{QR},$$

or, according to Fig. 114,

$$\tan \epsilon = \frac{0 \cdot 852 - 0 \cdot 710}{1} = 0 \cdot 142,$$

also

$$\tan \alpha = \frac{NR - FS}{1} = 0 \cdot 4725;$$

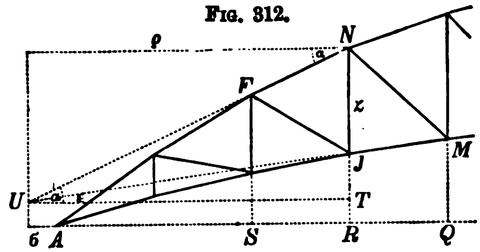
therefore

$$\rho = \frac{1 \cdot 065}{0 \cdot 4725 - 0 \cdot 142} = 3 \cdot 22.$$

The horizontal distance of U from A is therefore

$$\sigma = 3 \cdot 22 - 3 = 0 \cdot 22.$$

FIG. 312.



Braced Arch (§ 22).

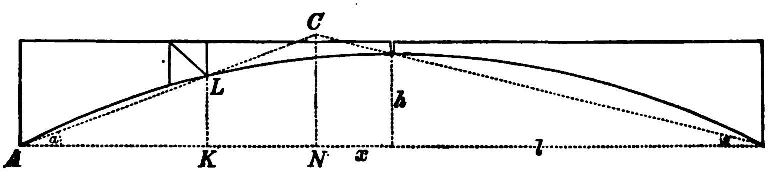
The position of the loading boundary at C (Fig. 174) can be determined from Fig. 313 by the equations

$$\mathrm{CN} = (l + x) \tan \epsilon = (l - x) \tan \alpha,$$

or

$$x = l\left(\frac{\tan \alpha - \tan \epsilon}{\tan \alpha + \tan \epsilon}\right).$$

FIG. 313.



Putting (from Fig. 174),

$$l = 20, \qquad \tan \epsilon = \frac{h}{l} = \frac{5}{20} = 0 \cdot 25,$$

and

$$\tan \alpha = \frac{\mathrm{LK}}{\mathrm{AK}} = \frac{3 \cdot 75}{10} = 0 \cdot 375,$$

$$x = 20\left(\frac{0 \cdot 375 - 0 \cdot 25}{0 \cdot 375 + 0 \cdot 25}\right) = 4.$$

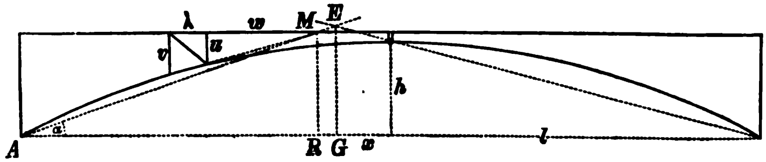
The loading boundary through F (Fig. 192) can be found from the same equation by substituting for $\tan a = \frac{5\cdot 5}{8} = 0\cdot 6875$, thus:

$$x = 20\left(\frac{0\cdot 6875 - 0\cdot 25}{0\cdot 6875 + 0\cdot 25}\right) = 9\cdot 3\dot{3}.$$

The intersection of the chords of the parabola with the horizontal upper *boom* can be determined by the method adopted for parabolic trusses (page 236). Thus the position of the point M in Figs. 180 and 182 can be found from the equation (Fig. 314),

$$\frac{u}{w} = \frac{v}{w + \lambda}, \text{ or } w = \frac{\lambda u}{v - u},$$

Fig. 314.



but from Fig. 173,

$$\lambda = 2, \quad u = 1\cdot 75, \quad v = 2\cdot 3,$$

therefore,

$$w = \frac{2 \times 1\cdot 75}{2\cdot 3 - 1\cdot 75} = 6\cdot 36,$$

and

$$\text{AR} = w + 5\lambda = 16\cdot 36.$$

The position of the loading boundary through E can now be obtained by putting $\tan a = \frac{\text{MR}}{\text{AR}} = \frac{5\cdot 5}{16\cdot 36} = 0\cdot 336$ in the previous equation, thus:

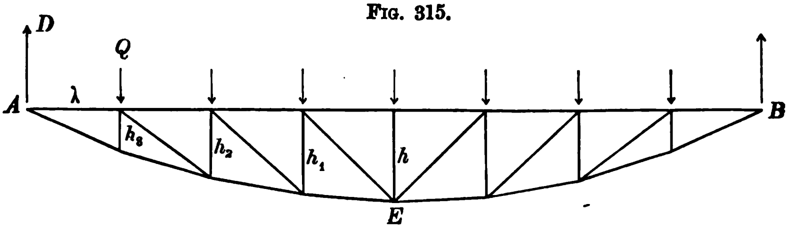
$$x = 20\left(\frac{0\cdot 336 - 0\cdot 25}{0\cdot 336 + 0\cdot 25}\right) = 2\cdot 9.$$

§ 37.—Application of the Method of Moments to find the Form a Structure should have in order that it may fulfil given Conditions.

In all the preceding examples the form of the structure was given, and the method of moments was employed to find the stresses produced in the various bars by the application of known loads. It will now be shown how the same method may be employed to determine the form a structure should have in order that it may fulfil certain given conditions.

The form and dimensions of the parabolic girder calculated in § 6 were given, and in determining the stresses it was found that when the bridge was fully loaded the stress in all the diagonals vanished, a property which was explained and found to belong to all parabolic girders in the subsequent "theory of parabolic girders," § 8. The operation could, however, be reversed, and it might be required to ascertain what form must be given to a girder in order that it may possess the above property. If for instance the span, the number of bays, and the depth of the girder are given, from which (Fig. 315) the points A, E, B, and likewise the positions of the loads

Fig. 315.



on the upper boom are determined, the only unknowns, if the girder be symmetrical, are the heights h_1, h_2, h_3.

h_1 can be found by forming the equation of moments for the part of the girder shown in Fig. 316, the turning-point being O_1, the intersection of the directions of the stresses X and Z. Now by the conditions this point must have such a position that $Y = o$. Hence the equation

$$0 = -D x_1 + Q \{ (x_1 + \lambda) + (x_1 + 2\lambda) + (x_1 + 3\lambda) \} .$$

But

$$D = \frac{7 Q}{2},$$

substituting and solving for x_1

$$x_1 = 12 \lambda.$$

The height h_1 can now be calculated from the equation

$$\frac{h_1}{h} = \frac{x_1 + 3 \lambda}{x_1 + 4 \lambda} = \frac{15}{16}.$$

In a similar manner the following equation is obtained from Fig. 317:

$$0 = - D x_2 + Q \{ (x_2 + \lambda) + (x_2 + 2 \lambda) \}.$$

FIG. 316.

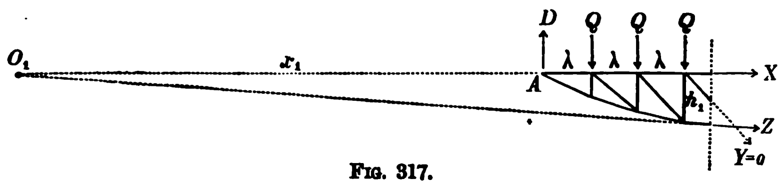
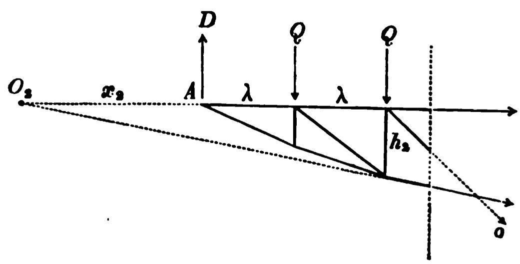


FIG. 317.



And again putting $D = \frac{7Q}{2}$ and solving for x_2,

$$x_2 = 2 \lambda;$$

whence h_2 can be found from the equation,

$$\frac{h_2}{h_1} = \frac{x_2 + 2 \lambda}{x_2 + 3 \lambda} = \frac{4}{5};$$

or replacing h_1 by its value in terms of h,

$$h_2 = \tfrac{4}{5} \cdot \tfrac{15}{16} h = \tfrac{3}{4} h.$$

Lastly, to determine h_3 (Fig. 318),

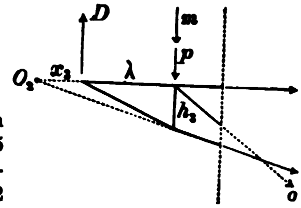
$$0 = -D x_3 + Q(x_3 + \lambda), \qquad \text{or} \quad \tfrac{7}{2} Q x_3 = Q(x_3 + \lambda)$$

$$x_3 = \tfrac{2}{5}\lambda$$

$$\frac{h_3}{h_2} = \frac{x_3 + \lambda}{x_3 + 2\lambda} = \frac{7}{12}$$

$$h_3 = \tfrac{7}{12} h_2 = \tfrac{7}{16} h.$$

FIG. 318.



As an example put $h = 2$ metres, as in Fig. 21, then $h_1 = 1{\cdot}875$ metre, $h_2 = 1{\cdot}5$ metre and $h_3 = 0{\cdot}875$ metre, or the dimensions given in Fig. 21. If further, $\lambda = 2$ metres; $x_1 = 24$ metres, $x_2 = 4$ metres, and $x_3 = 0{\cdot}8$ metre; thus assigning to O_1, O_2, and O_3 the same positions that were obtained graphically in § 6, and by calculation in § 37.

§ 38.—GIRDER IN WHICH THE MINIMUM STRESS IN THE DIAGONALS IS ZERO. (Schwedler's Girder.)

If the symmetrical parabolic girder of Fig. 35 be compared with the symmetrical girder with parallel booms of Fig. 69, it will be observed that in the first the maximum and the minimum stress in each diagonal are numerically equal but of opposite signs, whereas in the second, if the diagonals are inclined upwards from the centre towards the ends, the maximum stress has the largest numerical value, and the minimum stress is positive in all the bays except in those near the centre (and in fact the minimum stress in these latter diagonals would also become positive if the permanent load were sufficiently large in comparison to the moving load). It might therefore be expected that there exists an intermediate form of girder in which the minimum stress in all the diagonals is nothing.

This form of girder will now be found; it will be assumed that the number of bays is eight (as in § 37), the depth of the girder at the centre $= h$, the length of a bay $= \lambda$, and the span $= 8\lambda$. If the girder be symmetrical h_1, h_2, h_3 are the only dimensions required to determine its form (Fig. 319).

To find h_1 the girder must be so loaded that the stress Y is a minimum. This condition of loading is given in Fig. 319. Taking moments for the part of the girder shown in Fig. 320

about the point O_1, the intersection of the direction of the stresses X and Z, and putting Y = o,

$$0 = -Dx_1 + (p+m)\{(x_1+\lambda)+(x_1+2\lambda)+(x_1+3\lambda)\};$$

but

$$D = \tfrac{7}{2}p + m(\tfrac{7}{8}+\tfrac{6}{8}+\tfrac{5}{8}),$$

substituting and solving for x_1

$$x_1 = \frac{24(p+m)\lambda}{2p-3m}. \qquad [1]$$

Now h_1 can be found from the equation

$$\frac{h_1}{h} = \frac{x_1+3\lambda}{x_1+4\lambda},$$

or replacing x_1 by its value

$$\frac{h_1}{h} = \frac{30p+15m}{32p+12m}. \qquad [1A]$$

FIG. 319.

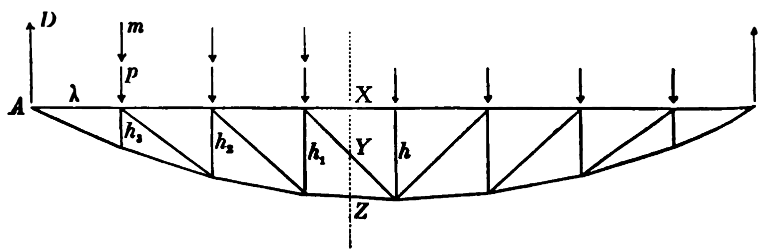
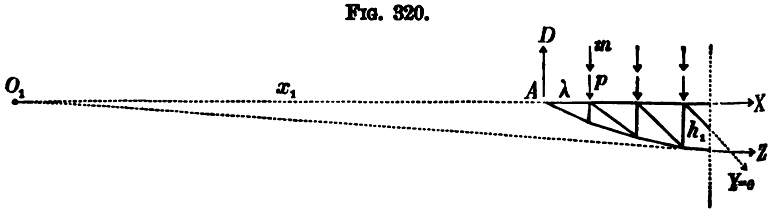


FIG. 320.



In the second bay (counted from the centre) the stress in the diagonal will be a minimum when the girder is loaded, as shown in Fig. 321, and

$$D = \tfrac{7}{2}p + m(\tfrac{7}{8}+\tfrac{6}{8}).$$

Taking moments about O_2 for the part of the girder shown in Fig. 322, and remembering that the stress in the diagonal is zero,

$$0 = -\mathrm{D}\,x_2 + (p+m)\,\{(x_2+\lambda)+(x_2+2\lambda)\};$$

whence

$$x_2 = \frac{8\,(p+m)\,\lambda}{4p-m}; \qquad [2]$$

FIG. 321.

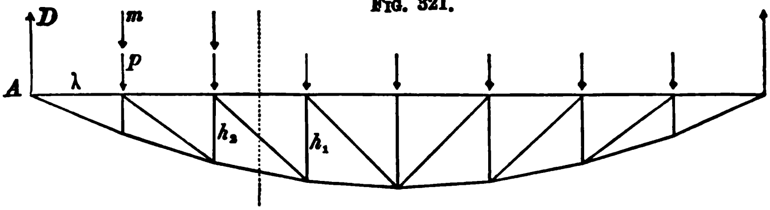
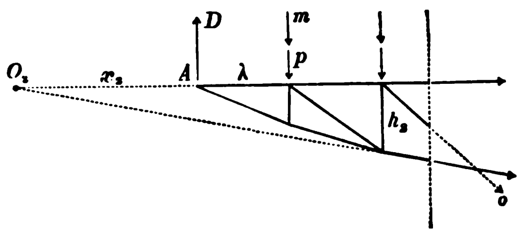


FIG. 322.



but h_2 can be found from the equation

$$\frac{h_2}{h_1} = \frac{x_2+2\lambda}{x_2+3\lambda},$$

or substituting for x_2,

$$\frac{h_2}{h_1} = \frac{16\,p+6\,m}{20\,p+5\,m}. \qquad [2\mathrm{A}]$$

Similarly x_3 and h_3 can be obtained as follows (Figs. 323 and 324):

$$0 = -\mathrm{D}\,x_3 + (p+m)(x_3+\lambda)$$

$$\mathrm{D} = \tfrac{7}{2}\,p + \tfrac{7}{8}\,m$$

$$x_3 = \frac{8\,(p+m)\,\lambda}{20\,p-m}. \qquad [3]$$

$$\frac{h_3}{h_2} = \frac{x_3+\lambda}{x_3+2\lambda} = \frac{28\,p+7\,m}{48\,p+6\,m}. \qquad [3\mathrm{A}]$$

By means of the foregoing equations the effect of altering the proportion between the permanent and moving loads can be studied.

Thus if $m = 0$ the girder is parabolic, as is shown by the following results:

$$x_1 = 12\,\lambda, \quad x_2 = 2\,\lambda, \quad x_3 = 0{\cdot}4\,\lambda$$

$$\frac{h_1}{h} = \frac{15}{16}, \quad \frac{h_2}{h_1} = \frac{4}{5}, \quad \frac{h_3}{h_2} = \frac{7}{12},$$

which are the values obtained in the last example, § 37.

FIG. 323.

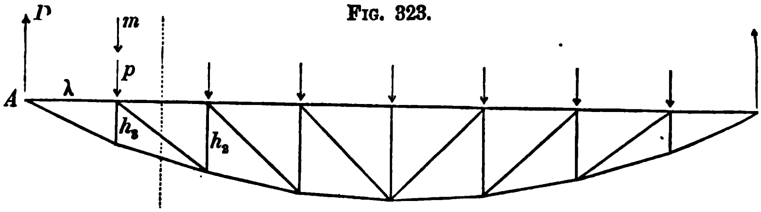
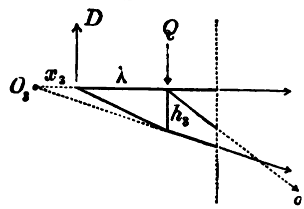


FIG. 324.



Suppose $\dfrac{m}{p} = \dfrac{1}{2}$,

then

$$\frac{x_1}{\lambda} = 72, \quad \frac{x_2}{\lambda} = \frac{24}{7}, \quad \frac{x_3}{\lambda} = \frac{24}{39},$$

and

$$\frac{h_1}{h} = \frac{75}{76}, \quad \frac{h_2}{h_1} = \frac{38}{45}, \quad \frac{h_3}{h_2} = \frac{21}{34}.$$

Thus, if, for example, $h = 1$,

$$h_1 = \tfrac{75}{76}, \quad h_2 = \tfrac{5}{6}, \quad h_3 = \tfrac{35}{68}.$$

These heights can be plotted above the horizontal A B, as in Fig. 325, or one-half can be plotted above and the other half beneath, as shown in Fig. 326. In

FIG. 325.

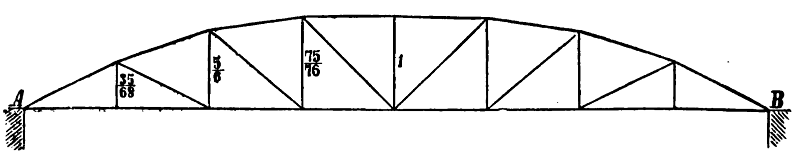
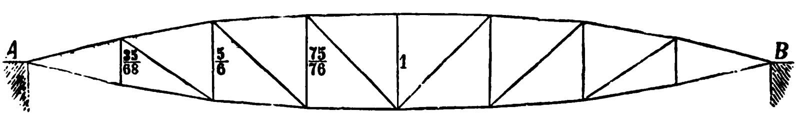


FIG. 326.

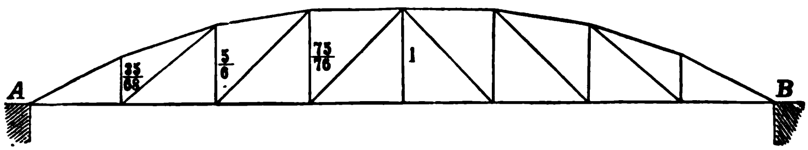


both cases tension alone will occur in the diagonals. (If, however, the diagonals were inclined the other way, as in Fig. 327, they would always be in compression, and the maximum stress would be nothing.)

Again, if $\dfrac{m}{p} = \dfrac{2}{3}$ it will be found that $x_1 = \infty$, and $\dfrac{h_1}{h} = 1$, and when $\dfrac{m}{p} > \dfrac{2}{3}$,

x_1 becomes negative, and $h_1 > h$. In this last case the depth of the girder in the centre would be less than it is in the adjoining bays. For other obvious reasons it would not be advisable to construct the girder thus; the conditions cannot therefore be complied with in this case.

FIG. 327.



Further, let $\frac{m}{p} = \frac{4}{3}$, then from equation 1A, $\frac{h_1}{h} = \frac{50}{47}$ but, instead of this, $\frac{h_1}{h} = 1$ would be taken. The equations for h_2 and h_3 are still applicable. For from equation 2A, $\frac{h_2}{h_1} = \frac{9}{10}$, or $h_2 = 0 \cdot 9\, h$; and from equation 3A, $\frac{h_3}{h_2} = \frac{2}{3}$; whence $h_3 = \frac{2}{3} \times 0 \cdot 9\, h$, or $h_3 = 0 \cdot 6\, h$.

Thus if the girder be 8 metres deep and the span 64 metres, the permanent load $p = 12,000$ kilos., and the moving load $m = 16,000$ kilos., the dimensions

FIG. 328.

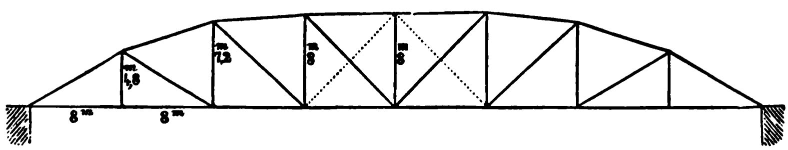


FIG. 329.

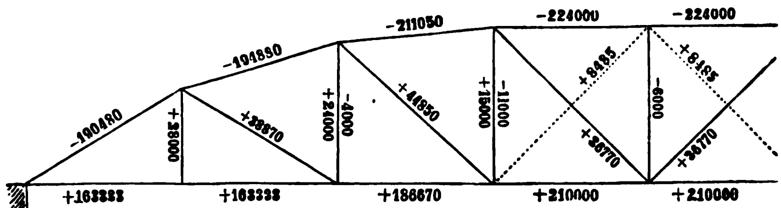
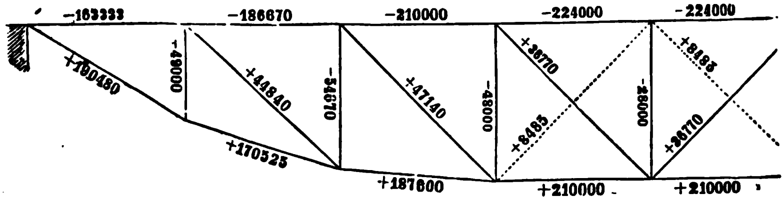


FIG. 330.



obtained would be those given in Fig. 328. The stresses can be found in the manner described in the Second Chapter, and are given in Fig. 329. If the

level of the rails is above instead of below, the girder would be of the form shown in Fig. 330.

The stresses in the verticals given for both girders are computed on the assumption that the whole of the permanent load is concentrated at the joints carrying the track. If, however, it is supposed that half the permanent load is applied to the upper joints, and the other half to the lower ones (in accordance with § 12), -6000 kilos. must be added to the stress in all the verticals of Fig. 329, and the stresses in these verticals, from the centre outwards, will become:

$$-12000, \begin{cases} -17000 \\ +9000 \end{cases}, \begin{cases} -10000 \\ +18000 \end{cases}, +22000.$$

In Fig. 330, however, $+6000$ kilos. must be added to the stresses in the verticals, thus:

$$-22000, \quad -42000, \quad -48670, \quad -43000.$$

When $\frac{m}{p} = 4$, it appears that $x_2 = \infty$ (equation 2), and when $\frac{m}{p} > 4$, x_2' is negative, and $h_2 > h_1$. In this case, therefore, it will be necessary to make h_2 as well as h_1 equal to h, and the four central bays will be rectangular. If, for instance, $p = 1000$ kilos., $m = 5000$ kilos., and $\lambda = h = 2$ metres (as in the girder calculated in § 6), it will be found that $x_3 = 6 \cdot 4$ metres, and $h_3 = 1 \cdot 615$ metre. Fig. 331 therefore represents the form of the girder and Fig. 332 gives the stresses in it, those in the four central bays coinciding with those given in Fig. 61.

Fig. 331.

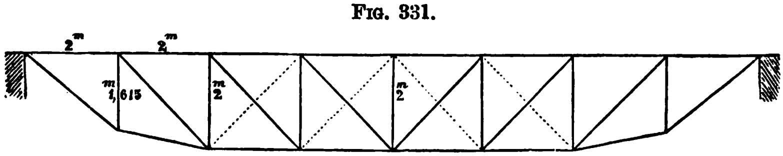
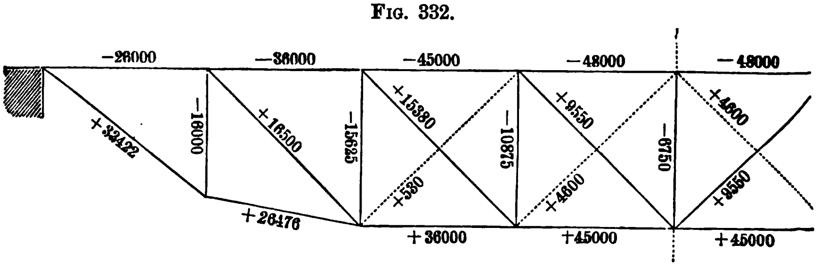


Fig. 332.



Lastly, if $m > 20p$, x_3 is negative (equation 3), and $h_3 > h_2$. As it cannot be considered advisable to make the depth of the girder diminish towards the centre, in this case the above conditions cannot be fulfilled in any bay.

§ 39.—Girder having the Maximum Stress in its Diagonals equal.

The problem discussed in § 38, namely, that the minimum stress in all the diagonals should be zero, is only a special case of the following: To determine the form of a girder in which the maximum stress in the diagonals is equal to some given quantity.

Since by turning the girder upside down all the stresses change their sign—and consequently the maximum stress becomes the minimum, and *vice versâ*—it is of no consequence whether the maximum or the minimum stresses be assumed equal to a given quantity. But as the minima stresses are usually negative, it is perhaps as well to make the assumption with regard to the maxima stresses. This will now be done, and it will be supposed that the girder is symmetrical, that there are eight bays, and that the maximum stress in the diagonals is equal to Y.

Fig. 333.

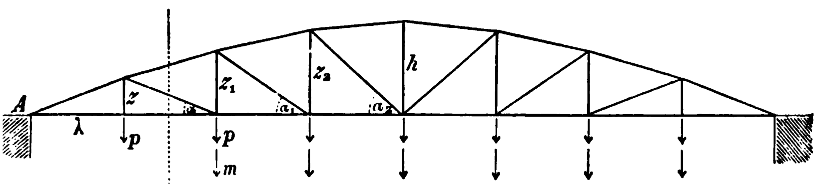
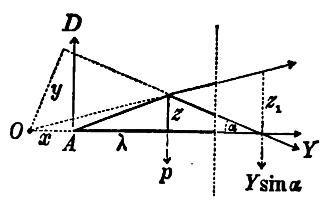


Fig. 334.



The stress in the diagonal of the second bay will be a maximum when the girder is loaded, as shown in Fig. 333, and the equation of moments (Fig. 334) is

$$0 = Yy - Dx + p(x + \lambda).$$

Or by substituting for y its value:

$$y = (x + 2\lambda)\sin\alpha,$$

$$0 = Y(x + 2\lambda)\sin\alpha - Dx + p(x + \lambda).$$

(This equation could also have been arrived at by resolving Y into its horizontal and vertical components, when the moment of the horizontal component would have vanished.)

Now,

$$D = \tfrac{7}{2}p + (\tfrac{1}{8} + \tfrac{2}{8} + \tfrac{3}{8} + \tfrac{4}{8} + \tfrac{5}{8} + \tfrac{6}{8})m = 3\cdot5\,p + 2\cdot625\,m;$$

therefore substituting and solving for x,

$$\frac{x}{\lambda} = \frac{p + 2\,Y \sin \alpha}{2\cdot5\,p + 2\cdot625\,m - Y \sin \alpha}. \qquad [1]$$

x_1 can be determined in the same manner from Fig. 335 (the five joints to the right of the section line being loaded) thus:

$$0 = Y(x_1 + 3\lambda) \sin \alpha_1 - D_1 x_1 + p\,\{(x_1 + \lambda) + (x_1 + 2\lambda)\}$$

$$D_1 = 3\cdot5\,p + 1\cdot875\,m.$$

$$\frac{x_1}{\lambda} = \frac{3p + 3\,Y \sin \alpha_1}{1\cdot5\,p + 1\cdot875\,m - Y \sin \alpha_1}. \qquad [2]$$

FIG. 335.

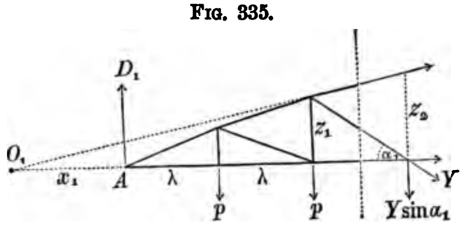
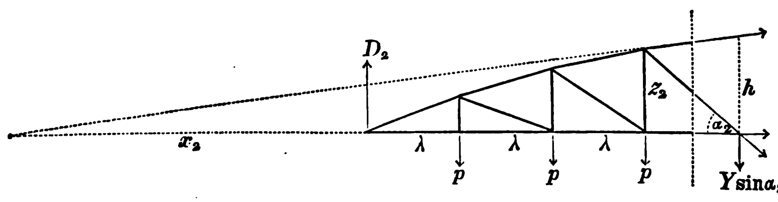


FIG. 336.



And lastly, to find x_2 (Fig. 336) (the four joints to the right of the section line being loaded):

$$0 = Y(x_2 + 4\lambda) \sin \alpha_2 - D_2 x_2 + p\,\{(x_2 + \lambda) + (x_2 + 2\lambda) + (x_2 + 3\lambda)\}$$

$$D_2 = 3\cdot5\,p + 1\cdot25\,m.$$

$$\frac{x_2}{\lambda} = \frac{6p + 4\,Y \sin \alpha_2}{0\cdot5\,p + 1\cdot25\,m - Y \sin \alpha_2}. \qquad [3]$$

The form of the girder can be obtained from these three equations in the following manner. Some value being assumed for z, $\sin \alpha$ can be found from the equation

$$\sin \alpha = \frac{z}{\sqrt{z^2 + \lambda^2}},$$

and then x is known from equation 1, whence z_1 (Fig. 334) can be determined from the equation

$$\frac{z_1}{z} = \frac{x + 2\lambda}{x + \lambda}. \qquad [4]$$

x_1 can now be found from equation 2 by substituting

$$\sin \alpha_1 = \frac{z_1}{\sqrt{z_1^2 + \lambda^2}},$$

and therefore z_2 can be obtained from (Fig. 335)

$$\frac{z_2}{z_1} = \frac{x_1 + 3\lambda}{x_1 + 2\lambda}. \qquad [5]$$

Likewise x_2 can be obtained from equation 3 by substituting

$$\sin \alpha_2 = \frac{z_2}{\sqrt{z_2^2 + \lambda^2}},$$

and h can then be found from the following equation (Fig. 336):

$$\frac{h}{z_2} = \frac{x_2 + 4\lambda}{x_2 + 3\lambda}. \qquad [6]$$

As an example, suppose that in the girder of § 6 it was wished to diminish the greatest compression in the diagonals, thereby increasing the greatest tension. For instance, let it be assumed that the maximum stress in every diagonal should be 8000 kilos., then in the above equations, $m = 5000$, $p = 1000$, and $Y = +8000$. Assuming that $z = \frac{1}{2}$ (taking the length of a bay as unity), the following values are obtained by following the steps indicated above; $\sin \alpha = 0 \cdot 447$, $x = 0 \cdot 677$, $z_1 = 0 \cdot 798$, $\sin \alpha_1 = 0 \cdot 625$, $x_1 = 3 \cdot 06$, $z_2 = 0 \cdot 958$, $\sin \alpha_2 = 0 \cdot 693$, $x_2 = 23 \cdot 28$, $h = 0 \cdot 996$. The form of the girder obtained is shown in Fig. 337, and the stresses in the various bars are given in Fig. 338.

As another example, let $Y = 0$, the above equations then give the form of a girder the diagonals of which are always in compression, and if at the same time $\frac{m}{p} = \frac{1}{2}$, it will be found that

$$\frac{x}{\lambda} = \frac{16}{61}, \quad \frac{x_1}{\lambda} = \frac{16}{13}, \quad \frac{x_2}{\lambda} = \frac{16}{3},$$

and

$$\frac{z_1}{z} = \frac{138}{77}, \quad \frac{z_2}{z_1} = \frac{55}{42}, \quad \frac{h}{z_2} = \frac{28}{25}.$$

By multiplying these three last equations together,

$$\frac{h}{z} = \frac{28}{25} \cdot \frac{55}{42} \cdot \frac{138}{77} = \frac{92}{35}.$$

From this equation z can be found as soon as some value is given to h. If, for instance, $h = 1$:

$$z = \frac{35}{92}, \quad z_1 = \frac{138}{77} \cdot \frac{35}{92} = \frac{15}{22}, \quad z_2 = \frac{25}{28} \quad \text{(Fig. 339.)}$$

FIG. 337.

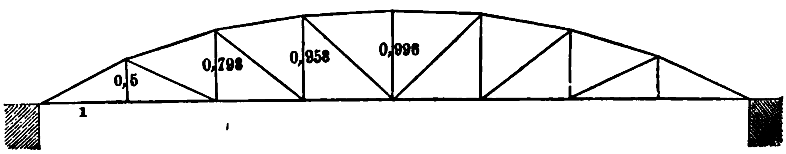


FIG. 338.

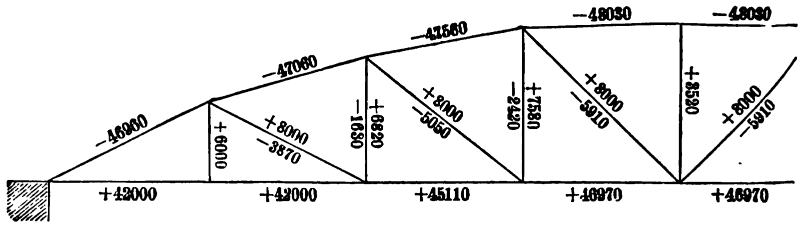
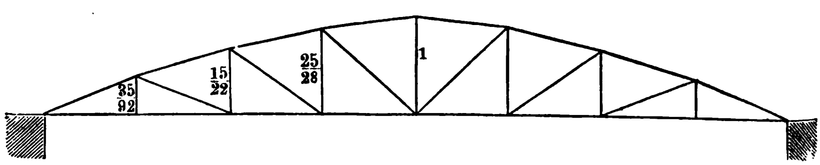


FIG. 339.



In the girders shown in Fig. 327 and Fig. 339 the diagonals are always in compression. If these girders were reversed the diagonals would always be in tension. It appears, therefore, that there are two solutions to the problem of § 38, and that the form of the girder obtained depends on the direction in which the diagonals are inclined. If the direction be that shown in Fig. 327, the conditions can only be complied with, it was seen, to a certain extent when the ratio $\frac{m}{p}$ lies between $\frac{2}{3}$ and 20, and not at all when $\frac{m}{p} > 20$. But with the direction of the diagonals chosen in this § a girder can always be designed meeting the imposed conditions. Even in the extreme case, when $\frac{m}{p} = \infty$ or $p = 0$, the *above equations* give results that can be practically applied. Then x_1, x_2, x_3,

all become zero, and the truss takes the triangular form shown in Fig. 340. If, on the contrary, $\frac{p}{m} = \infty$ or $m = 0$, the limiting form is the parabolic girder. It is obvious that it makes no difference whether the ordinates be plotted above or below the horizontal through the abutments, or whether a part be placed

FIG. 340.

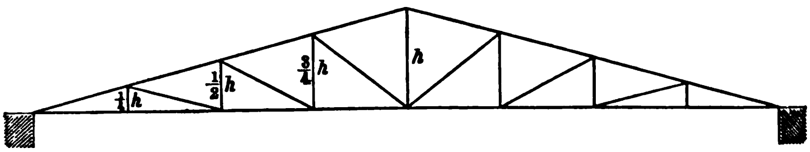


FIG. 341.

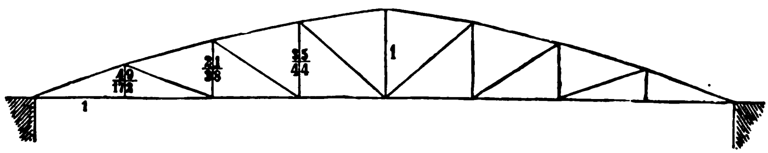


FIG. 342.

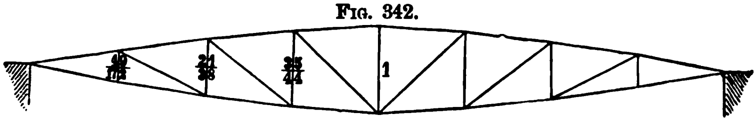


FIG. 343.

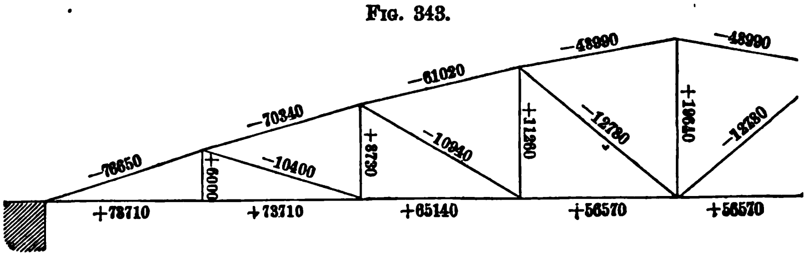
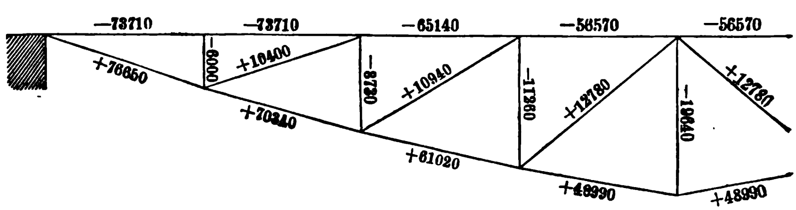


FIG. 344.



above and the remainder below. The diagonals in the girder thus obtained will always be in compression, and if it be reversed the diagonals will always be in tension.

Again, let $Y = 0$, $p = 1000$ kilos., $m = 5000$ kilos., and $h = \lambda = 1$, then by solving equations 1 to 6 the girder given in Fig. 341 is obtained, and of which Fig. 342 is a variation. By comparing this girder with that of Fig. 338, in which the maxima stresses in the diagonals are equal, with the Schwedler's girder of Fig. 332, and with the parabolic girder of Fig. 39, the influence the form of the girder has on the stresses will become apparent.

The stresses in the girder of Fig. 341 are given in Fig. 343, and by multiplying these stresses by -1, those for the girder of Fig. 344 are obtained, in which the diagonals are always in tension. In either case it is assumed that both the moving and permanent loads are applied to the joints situated in the horizontal boom. If half the permanent load is concentrated on the upper joints and the remainder on the lower joints, -500 must be added to the stresses in the verticals of Fig. 343 and $+500$ to those of Fig. 344.

If in Fig. 333 the loads were applied to the upper extremity of the verticals instead of to the lower, the values of $Y \sin \alpha$ and $Y \sin \alpha_1$, which appear in equations 1 and 2, taken negatively, would each be the stress in the vertical to the right of the corresponding diagonal. The above equations can therefore be employed to determine the form of a girder in which the greatest compression in the verticals is equal to some given quantity, by making the vertically resolved part of Y equal to this quantity.

§ 40.—Girder in which the Stresses in the Bow are the same throughout. (Pauli's Girder.)

In the symmetrical parabolic girder of Fig. 34 the stresses in the bow increase from the centre towards the ends; but in the girder of Fig. 70, with parallel booms, the stresses in the booms, on the contrary, diminished from the centre towards the ends. It is evident, therefore, that some intermediate form must exist, in which the stresses in the bow are equal throughout. It has been seen that the stresses in the booms are greatest when the girder is fully loaded; the moving load need not, therefore, be separated from the permanent load. Let Q be the total load on each loaded joint (Fig. 345), then the girder having eight bays, the reaction at each abutment will be $D = \dfrac{7\,Q}{2} = 3 \cdot 5\,Q$.

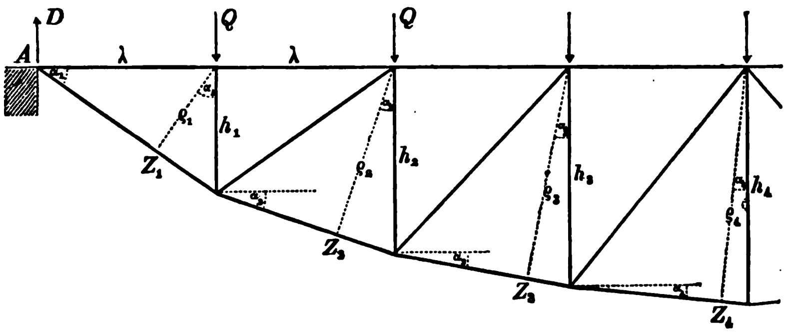
The stress in the bow in any one of the bays situated to the left of the centre can be found from the equation of moments, formed with reference to the part of the girder to the left of a

vertical section cutting through the bay in question, the intersection of the diagonal and the top boom being the turning-point.

Thus, from Fig. 345, are obtained the following equations to find the stresses Z_1, Z_2, Z_3, and Z_4:

$$
\begin{aligned}
Z_1\,\rho_1 &= D\,\lambda \\
Z_2\,\rho_2 &= D\,2\,\lambda - Q\,\lambda \\
Z_3\,\rho_3 &= D\,3\,\lambda - Q\,(2\,\lambda + \lambda) \\
Z_4\,\rho_4 &= D\,4\,\lambda - Q\,(3\,\lambda + 2\,\lambda + \lambda).
\end{aligned}
$$

FIG. 345.



Let $M_1 \ldots M_4$ be the sum of the moments on the right-hand side of these equations, then, after substituting for D,

$$M_1 = 3{\cdot}5\,Q\,\lambda, \qquad M_2 = 6\,Q\,\lambda, \qquad M_3 = 7{\cdot}5\,Q\,\lambda, \qquad M_4 = 8\,Q\,\lambda.$$

Now, according to the conditions, the stresses $Z_1 \ldots Z_4$ must be equal, say, to Z: hence

$$Z\,\rho_1 = M_1, \qquad Z\,\rho_2 = M_2, \qquad Z\,\rho_3 = M_3, \qquad Z\,\rho_4 = M_4.$$

Suppose, for instance, that $\lambda = 1$ and $Q = 6000$ kilos., then $M_1 = 21000$, $M_2 = 36000$, $M_3 = 45000$, $M_4 = 48000$. If, therefore, the stress Z is to be equal to 36000 kilos. throughout the bow,

$$\rho_1 = \frac{21000}{36000} = \frac{7}{12}, \quad \rho_2 = \frac{36000}{36000} = 1, \quad \rho_3 = \frac{45000}{36000} = \frac{5}{4}, \quad \rho_4 = \frac{48000}{36000} = \frac{4}{3}.$$

To find the form of the girder by construction, describe circles with radii $\rho_1 \ldots \rho_4$ from the corresponding turning-points as centres, and draw the lower boom in each bay a tangent to its circle. Starting from the abutment A, Fig. 346 is thus obtained, and the depths $h_1 \ldots h_4$ can be then found by measurement.

The form of the girder can also be obtained by calculation, instead of by construction, in the following manner:—From Fig. 345 two expressions can be obtained for $\cos \alpha_1$, $\cos \alpha_2$, $\cos \alpha_3$, and $\cos \alpha_4$, and $h_1 \ldots h_4$ can be found by equating these expressions. Thus, in the first bay,

$$\frac{\rho_1}{h_1} = \frac{\lambda}{\sqrt{\lambda^2 + h_1^2}} = \cos \alpha_1; \qquad [1]$$

or,

$$\frac{h_1}{\lambda} = \frac{\rho_1}{\sqrt{\lambda^2 - \rho_1^2}}; \qquad [I.]$$

and h_1 can be obtained from this equation, for $\rho_1 = \frac{M_1}{Z}$ is known.

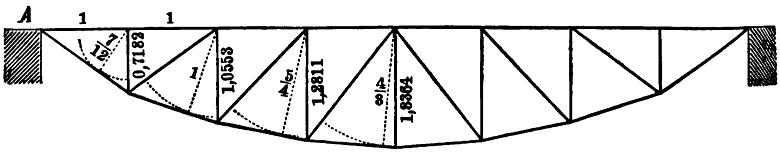
For the second bay,

$$\frac{\rho_2}{h_2} = \frac{\lambda}{\sqrt{\lambda^2 + (h_2 - h_1)^2}} = \cos \alpha_2; \qquad [2]$$

solving this quadratic equation:

$$\frac{h_2}{\lambda} = \frac{\rho_2^2}{\lambda^2 - \rho_2^2}\left\{-\frac{h_1}{\lambda} + \sqrt{\left(\frac{\lambda}{\rho_2}\right)^2 + \left(\frac{h_1}{\rho_2}\right)^2 - 1}\right\}. \qquad [II.]$$

FIG. 346.



In a similar manner the following equations are obtained for the third and fourth bays:

$$\frac{\rho_3}{h_3} = \frac{\lambda}{\sqrt{\lambda^2 + (h_3 - h_2)^2}} = \cos \alpha_3. \qquad [3]$$

$$\frac{h_3}{\lambda} = \frac{\rho_3^2}{\lambda^2 - \rho_3^2}\left\{-\frac{h_2}{\lambda} + \sqrt{\left(\frac{\lambda}{\rho_3}\right)^2 + \left(\frac{h_2}{\rho_3}\right)^2 - 1}\right\}. \qquad [III.]$$

$$\frac{\rho_4}{h_4} = \frac{\lambda}{\sqrt{\lambda^2 + (h_4 - h_3)^2}} = \cos \alpha_4. \qquad [4]$$

$$\frac{h_4}{\lambda} = \frac{\rho_4^2}{\lambda^2 - \rho_4^2}\left\{-\frac{h_3}{\lambda} + \sqrt{\left(\frac{\lambda}{\rho_4}\right)^2 + \left(\frac{h_3}{\rho_4}\right)^2 - 1}\right\}. \qquad [IV.]$$

If the value chosen for Z were such that the lever-arm $\rho_1 = \frac{M_1}{Z} = \frac{D\lambda}{Z}$ became equal to λ, then, from equation I., it appears that $\frac{h_1}{\lambda} = \infty$. Therefore the conditions can no longer be complied with when $\frac{M_1}{Z} \geqq \lambda$, or $Z \leqq D$.

In the previous numerical example $\frac{M_1}{\lambda} = D = 21000$, if then Q remains equal to 6000 kilos., it would be impossible so to construct the girder that stress in the bow should everywhere be equal or less than 21000 kilos.

If Z has such a value that the lever-arm $\rho_2 = \frac{M_2}{Z}$ becomes equal to λ, equation II. takes the indeterminate form $\frac{h_2}{\lambda} = \infty \times 0$.

In such a case, h_2 must be found from equation 2, which, when λ is substituted for ρ_2, takes the form,

$$\frac{1}{h_2} = \frac{1}{\sqrt{\lambda^2 + (h_2 - h_1)^2}},$$

and solving for h_2

$$h_2 = \frac{\lambda^2 + h_1^2}{2\,h_1}, \qquad \text{[IIA.]}$$

or,

$$\frac{h_2}{\lambda} = \frac{1}{2}\left(\frac{\lambda}{h_1} + \frac{h_1}{\lambda}\right).$$

This case occurred in the preceding numerical example (Fig. 346), for Z was taken at 36000 kilos., and it was found that $\frac{M_2}{\lambda} = 36000$, therefore $\rho_2 = \frac{M_2}{Z} = \lambda = 1$. Hence, finding the value of h_1 from equation I., viz. $h_1 = 0{\cdot}7182$, and substituting in equation IIA.,

$$h_2 = \frac{1}{2}\left(\frac{1}{0{\cdot}7182} + 0{\cdot}7182\right) = 1{\cdot}0553;$$

then, from equation III., $h_3 = 1{\cdot}2811$; and finally $h_4 = 1{\cdot}3364$ from equation IV.

The same course would have to be pursued if, in any other bay, the lever-arm of Z became equal to λ.

For instance, if $Z = 48000$ kilos., and the other data remain the same as in the previous example, namely, $\lambda = 1$ and $Q = 6000$ kilos. (so that the moments are the

same, viz. $M_1 = 21000$, $M_2 = 36000$, $M_3 = 45000$, $M_4 = 48000$), the values obtained for the lever-arms are: (see Fig. 347)

$$\rho_1 = \frac{21}{48} = \frac{7}{16}, \quad \rho_2 = \frac{36}{48} = \frac{3}{4}, \quad \rho_3 = \frac{45}{48} = \frac{15}{16}, \quad \rho_4 = \frac{48}{48} = 1.$$

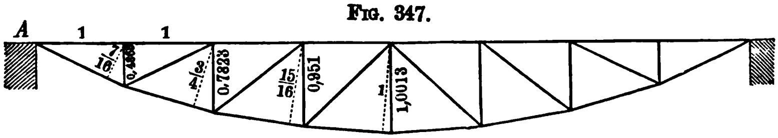
Then, from equations I., II., III.,

$$h_1 = 0{\cdot}4865,\ h_2 = 0{\cdot}7823,\ h_3 = 0{\cdot}951.$$

But equation IV. takes the indeterminate form, $\frac{h_4}{\lambda} = \infty \times 0$; h_4 must therefore be calculated from the equation:

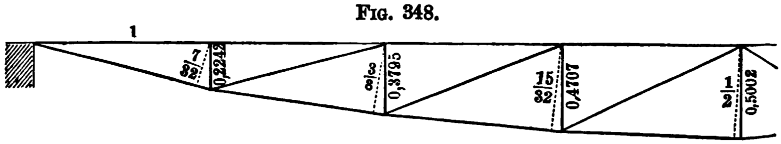
$$\frac{h_4}{\lambda} = \frac{1}{2}\left(\frac{\lambda}{h_3} + \frac{h_3}{\lambda}\right) = \frac{1}{2}\left(\frac{1}{0{\cdot}951} + 0{\cdot}951\right) = 1{\cdot}0013.$$

Fig. 347.



If the lever-arms $\rho_1 \ldots \rho_4$ are halved, the stress Z is doubled; but if at the same time Q is halved, Z does not alter. For instance, the dimensions of the girder given in Fig. 348 would apply when Q = 6000 kilos. and Z = 96000 kilos., or when Q = 3000 kilos. and Z = 48000 kilos., and generally in all cases when Z = 16 Q.

Fig. 348.



If the girder be reversed, the new stresses can be found by multiplying the old ones by − 1. By reversing, therefore, Fig. 348, the girder shown in Fig. 349 is obtained, in which the tension in the bow is everywhere the same, and is = + 48000 kilos. if Q = 3000 kilos.

The stresses given in Figs. 349 and 350 have been calculated on the supposition that the total load on each

loaded joint is 3000 kilos. (composed of a permanent load $\frac{p}{2} = \frac{1000}{2}$ kilos., and of a moving load, $\frac{m}{2} = \frac{5000}{2}$ kilos.). By imagining these two girders united to form Fig. 351, a girder is obtained in which the stress in the lower as well as in the upper bow is everywhere equal to 48,000 kilos. The ratio of the depth to span (1 : 8) and the loads (permanent load p = 1000 kilos., and moving load m = 5000 kilos.) are the same as those of the parabolic girder of § 6. The stresses in the horizontal booms in Fig. 351 destroy each other mutually; this boom is therefore omitted as being unnecessary. The

FIG. 349.

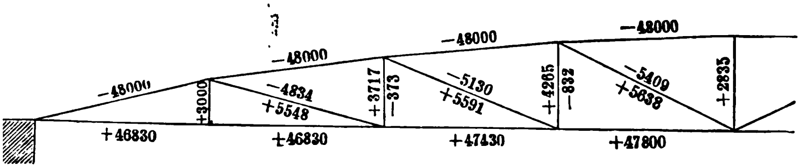


FIG. 350.

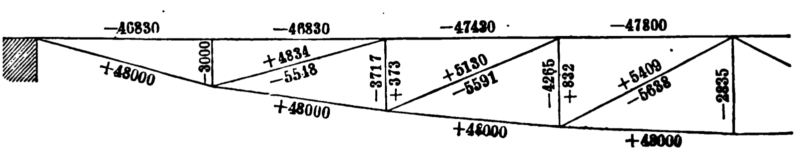
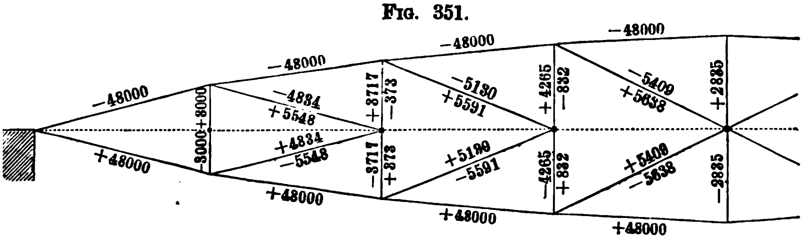


FIG. 351.



stresses in the verticals correspond to the assumption that the points of application of the permanent as well as of the moving load lie in the horizontal through the abutments.

If the girder is to have crossed diagonals, and not two half-diagonals meeting each other at the centre of the verticals, the depths h_1, $h_2 \ldots h_4$ will have to be slightly altered; but as

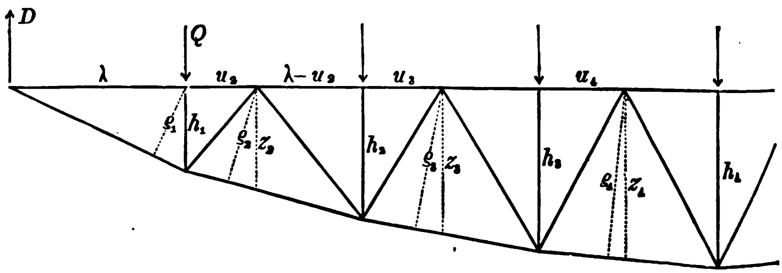
may be expected these alterations will be small, and, in fact, Fig. 351 is an approximate form.

To find the accurate form, the first step is to obtain the dimensions of the girder shown in Fig. 352 (which will become the lower half of the required girder), when the stress in the bow is throughout equal to the given stress.

The equations of moments are:

$$Z\rho_1 = D\lambda \qquad [5]$$
$$Z\rho_2 = D(\lambda + u_2) - Q u_2 \qquad [6]$$
$$Z\rho_3 = D(2\lambda + u_3) - Q\{(\lambda + u_3) + u_3\} \qquad [7]$$
$$Z\rho_4 = D(3\lambda + u_4) - Q\{(2\lambda + u_4) + (\lambda + u_4) + u_4\}. \qquad [8]$$

FIG. 352.



As there is no difference in Figs. 345 and 352 in the first bay, h_1 can be found from equation I., and by substituting for ρ_1 its value from equation 5.

$$\frac{h_1}{\lambda} = \frac{D}{\sqrt{Z^2 - D^2}}. \qquad [V.]$$

To determine u_2 and ρ_2, the following equations are obtained from Fig. 353:

$$\frac{h_1}{u_2} = \frac{h_2}{\lambda - u_2} = \tan \alpha_2,$$

$$\frac{z_2 - h_1}{u_2} = \frac{h_2 - h_1}{\lambda} = \tan \epsilon_2.$$

Solving the first equation for u_2,

$$u_2 = \frac{\lambda h_1}{h_1 + h_2}; \qquad [9]$$

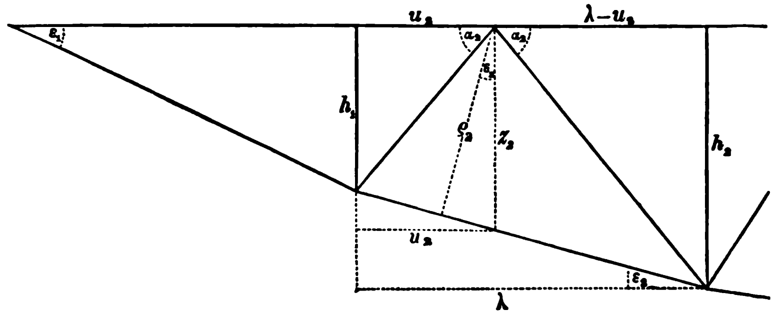
and finding z_2 from the second by substituting for u_2,

$$z_2 = \frac{2\,h_1\,h_2}{h_1 + h_2}.$$

Consequently,

$$\rho_2 = z_2 \cos \epsilon_2 = \frac{2\,h_1\,h_2 \cos \epsilon_2}{h_1 + h_2}. \qquad [10]$$

FIG. 353.



Substituting these values of u_2 and ρ_2 in equation 6, and solving for $\frac{h_2}{\lambda}$,

$$\frac{h_2}{\lambda} = \frac{(2\,D - Q)\,h_1}{2\,Z\,h_1 \cos \epsilon_2 - D\,\lambda}. \qquad [VI.]$$

From this equation h_2 can be found when h_1 has been calculated from equation V., and the angle ϵ_2 is known. But

$$\cos \epsilon_2 = \frac{\lambda}{\sqrt{\lambda^2 + (h_2 - h_1)^2}}; \qquad [11]$$

so that $\cos \epsilon_2$ depends on h_2. Now $\cos \epsilon_2$ is less than unity and greater than $\cos \epsilon_1$, therefore an approximation to h_2 can be found by assuming for $\cos \epsilon_2$ some value between these limits (or one of the limits may be first assumed, say the limit 1). The value thus found for h_2, substituted in equation 11, will give a nearer approximation to $\cos \epsilon_2$, from which a more accurate value of h_2 can be obtained, and so on until the required degree of accuracy has been arrived at.

For the third bay, equations similar to 9, 10, 11 are obtained —namely,

$$u_3 = \frac{\lambda h_2}{h_2 + h_3}. \qquad [12]$$

$$\rho_3 = \frac{2 h_2 h_3 \cos \epsilon_3}{h_2 + h_3}. \qquad [13]$$

$$\cos \epsilon_3 = \frac{\lambda}{\sqrt{\lambda^2 + (h_3 - h_2)^2}}. \qquad [14]$$

By substituting these values of u_3 and ρ_3 in equation 7, and solving for $\frac{h_3}{\lambda}$,

$$\frac{h_3}{\lambda} = \frac{3(D - Q) h_2}{2 Z h_2 \cos \epsilon_3 - (2 D - Q) \lambda}. \qquad [VII.]$$

Similarly, for the fourth bay,

$$u_4 = \frac{\lambda h_3}{h_3 + h_4}. \qquad [15]$$

$$\rho_4 = \frac{2 h_3 h_4 \cos \epsilon_4}{h_3 + h_4}. \qquad [16]$$

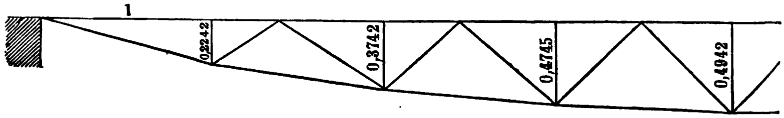
$$\cos \epsilon_4 = \frac{\lambda}{\sqrt{\lambda^2 + (h_4 - h_3)^2}}. \qquad [17]$$

and combining equations 15 and 16 with equation 8:

$$\frac{h_4}{\lambda} = \frac{(4 D - 6 Q) h_3}{2 Z h_3 \cos \epsilon_4 - 3(D - Q) \lambda}. \qquad [VIII.]$$

Again, assuming that $\lambda = 1$ and $Z = 16 Q$, as in the former numerical example (Fig. 348), the value of h_1, obtained from equation V., is 0·2242, as before. Putting this value of h_1 in equation VI., and assuming $\cos \epsilon_2 = 1$, h_2 is found to be 0·3661 as a first approximation. Equation 11 then gives 0·99 as a

FIG. 354.



nearer value for $\cos \epsilon_2$, which, substituted in equation VI., gives $h_2 = 0·3734$. A repetition of the same operation gives $\cos \epsilon_2 = 0·9889$, and $h_2 = 0·3742$ as a third approximation. In a similar manner, from equations VII. and 14, $h_3 = 0·4745$, and from equations VIII. and 17, $h_4 = 0·4942$. (See Fig. 354.)

The stresses given in Figs. 355 and 356 have been calculated on the supposition that the total load on a loaded joint is 3000 kilos, consisting of a permanent load, $\frac{p}{2} = \frac{1000}{2}$ kilos., and of a moving load, $\frac{m}{2} = \frac{5000}{2}$ kilos. (A comparison is thus obtained with the girders of Figs. 349 and 350.)

FIG. 355.

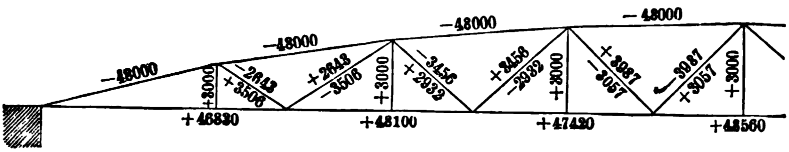


FIG. 356.

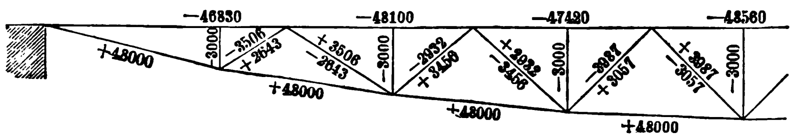
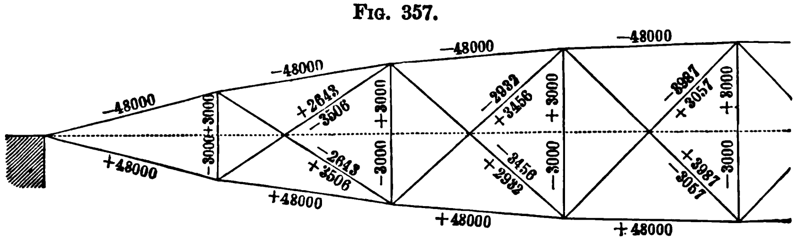


FIG. 357.



The combination of these two girders gives the girder shown in Fig. 357, in which the loading (permanent load $p = 1000$ kilos., and moving load $m = 5000$ kilos.) and the ratio of depth to span ($0 \cdot 9884 : 8$) are the same as those of the parabolic girder of § 6.

TWELFTH CHAPTER.

§ 41.—Determination of the Cross-Sectional Areas of the Bars in a Structure.

Each bar of a structure can be regarded as a bundle of rods firmly bound together, each rod having a cross-section equal to the unit of area. *To obtain, therefore, the number of units of area which the cross-section of any bar of a structure must contain, it is only necessary to divide the stress in the bar by what is considered to be the safe stress on a unit of area.*

So long as the stress is within the limits of elasticity, it can be considered safe. For instance, a wrought-iron rod whose cross-section has an area of 1 square millimetre can, on an average, have a stress of 15 kilos. applied to it without the limit of elasticity being exceeded; but any increase in the stress would produce a set. Fifteen kilos. per square millimetre can be, therefore, considered as the limit of safety; but in practice it is usual to allow only 6 or 8 kilos.* per square millimetre for wrought iron, and it is only in special cases, where the risk may be run, that the limit should be approached nearer.

Thus, to obtain the area of the cross-sections in square millimetres of the various parts of the structures considered in the preceding chapters (supposing them to be made of wrought iron), the calculated stresses in kilos. should be divided by 6.

For instance, the cross-sectional areas in square millimetres required for the braced girder of Fig. 61 are as follows:—

1. For the top boom:

   3500, 6000, 7500, 8000, 8000, 7500, 6000, 3500.

2. For the bottom boom:

   0, 3500, 6000, 7500, 7500, 6000, 3500, 0.

* *This is equivalent to 3·8 and 5·1 tons per sq. inch.*—Trans.

3. For the verticals:

4000, 3500, 2604, 1813, 1125, 1813, 2604, 3500, 4000.

4. For the diagonals:

4950, 3683, 2567, 1592, 1592, 2567, 3683, 4950.

5. For the counter-braces: *

88, 767, 767, 88.

It must, however, be carefully remembered that the resistance to compression cannot always be taken as equal to the resistance to tension; on the contrary, in many cases the resistance to compression is far less, when, in fact, the bar is a long column and is liable to fail by buckling. This point will be considered more fully in the sequel. (See "Resistance of Long Columns to Buckling.")

Further, it must be observed that these sectional areas are those due to the stresses in the main structure only; secondary structures may, however, be fused into the main structure, altering the sections accordingly.

The stresses obtained in the preceding chapters were calculated under the following assumptions:—1. That, except the reactions at the abutments, all the exterior forces acting on the structure are vertical forces. 2. That the joints are the only points of application of those forces.

To comply with these assumptions, it is generally necessary to add to the main structure intermediate bearers, which span the distance between the joints, and concentrate the load at the joints; and also a system of bracing, to resist the pressure of the wind or any other horizontal force.

Some of the bars of these secondary structures will run parallel, and close to some of the bars of the main structure. These parallel bars can either be left separate, or else, as it were, fused together. In the latter case, the stress, and therefrom the cross-section of the resulting bar, can be found by forming the algebraical sum of the stresses in each of the component bars.

* The second diagonal in the central bays is called a counter-brace.

§ 42.—Bracing required to resist the Pressure of the Wind and Horizontal Vibrations.

When a train passes over a bridge, horizontal forces are brought into play by the oscillation of the locomotive and of the carriages, and the pressure of the wind is also increased proportionately to the surface of the train exposed. To resist these forces, a system of horizontal bracing is introduced, producing in reality a horizontal girder, the booms of which are formed by those of the main girders.

The stresses in this horizontal girder can be obtained in the manner explained in the Third Chapter; for the main girders being always parallel to each other, this horizontal girder will always have parallel booms. The distinction made between the permanent load (uniformly distributed) and the moving load (at times unequally distributed) will also have to be made in this case, for the train, in progressing along the bridge, adds continually to the wind-pressure and to the horizontal oscillations.

It is not far from the truth to assert that the proportion between the permanent and moving load for the horizontal girder is the same as that for the main girder. Thus, if it happens that the width of the bridge is equal to the depth of the vertical girders, the stresses in the horizontal girder can at once be found, as explained in the Seventh Chapter.

There is only one point of dissimilarity between the two girders: in the vertical girder the loads always act downwards, but in the case of the horizontal girder they act sometimes on one side and sometimes on the other. The latter must, therefore, be constructed symmetrically with reference to the central line, and the stresses in the booms are to be marked with the sign $\pm$, for each part of the boom will have alternately to resist tension and compression of equal magnitude.

Further, if (as in Fig. 61) the diagonals are designed to resist tension only, counter-braces will have to be introduced in every bay, and not only in the four central bays, as in Fig. 61. There is, in fact, no difference in this case between a

brace and a counter-brace, for the bar that acts as a brace when the wind is blowing on one side of the bridge, will be a counter-brace when the wind is blowing in the opposite direction. Therefore, also, the greatest stresses in the diagonals of any bay are equal.

In Fig. 61 the line of railway is on a level with the upper boom; it is, therefore, possible (to prevent lateral distortion) to brace the two vertical girders together. The lower booms can also be braced together to form a second horizontal girder; and it can be assumed that by means of the transverse bracing the load on the two horizontal girders is equally distributed between them.

In continuation of § 41, let it be required to find the sectional areas of the various bars in these two horizontal girders.

The total vertical load on the bridge was assumed to be (see pp. 20 and 39)

$$p + w = 6000 \text{ kilos. per metre run.}$$

Supposing that the load on the horizontal girders is

$$p_1 + w_1 = 857 \text{ kilos. per metre run.}$$

the requisite sections for these girders can be found (assuming that the breadth and height of the bridge are equal) by multiplying those already obtained in § 41 by the ratio

$$\frac{p_1 + w_1}{p + w} = \frac{857}{6000} = \frac{1}{7},$$

remembering that the greatest sectional area obtained for any two symmetrical bars must alone be retained. Thus, for the left half of the girder the sectional areas, in square millimetres, are:

1. For the booms (either one side of the other),

[illegible]

2. For the cross-members (corresponding to the verticals),

[illegible]

3. For the diagonals (braces and counter-braces),

[illegible]

The stresses in the bars forming the transverse bracing can be obtained by simply resolving the horizontal force, 857 kilos.,* acting on each joint along them (supposing them to be placed at each joint). Thus in Fig. 358, if these braces can only resist tension, the stress in them is (alternately)

$$857 \times \sqrt{2} = +\ 1210 \text{ kilos.}$$

and the area of their cross-section will be

$$\frac{1210}{6} = 202 \text{ square millimetres.}$$

Each vertical in the main girders receives an increase of 857 kilos. compression, which should be added to the stresses already found, although this probably would not be done in practice. The sectional area already found for each vertical should, therefore, in this case be increased by $\frac{857}{6} = 143$ square millimetres (Fig. 359).

FIG. 358.

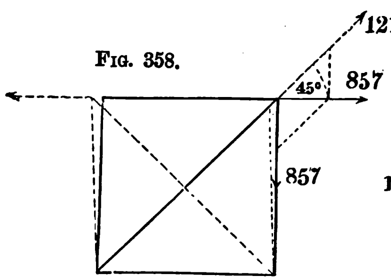
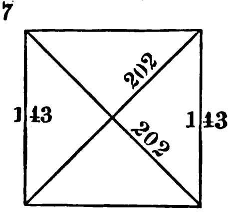


FIG. 359.



The numbers given in Figs. 360 and 361 express in square millimetres the cross-sectional area of each bar of the girder, and they are arrived at by combining the results just obtained. These can be considered as the final sections, if the intermediate bearers between the joints are constructed separately.

§ 43.—INTERMEDIATE BEARERS.

If the joints of the main structure are so far apart that they do not offer a sufficient number of points of support, it

* The actual horizontal force on each top joint is 2 × 857 kilos., according to the assumptions, but one-half of this is resisted by the top horizontal girder, leaving only 857 kilos. to be communicated to the lower horizontal girder.—TRANS.

becomes necessary (to fulfil the condition that the load on the main structure is concentrated at the joints) to introduce intermediate bearers that will span the distance between the joints, and transmit the load to them, and that will also furnish at the same time a sufficient number of points of support.

These intermediate trusses bear the same relation to the loads upon them that the main girder does to its loads; they can therefore be similarly constructed as a combination of bars.

If the number of points of support offered by one set of

Fig. 360.

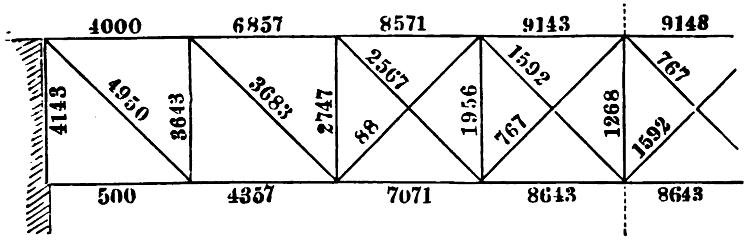
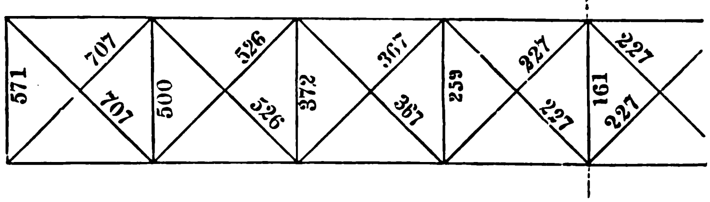


Fig. 361.



intermediate bearers be not sufficient, another set of a secondary order must be introduced. If even then the points of support are not near enough, a third order must be added, and so on until the required number of points of support is obtained. The triangles formed by the bracing of this last set may be so small that the material saved in the void spaces would not cover the extra expense for workmanship, and it then would be better to use a plate-web instead of the bracing.

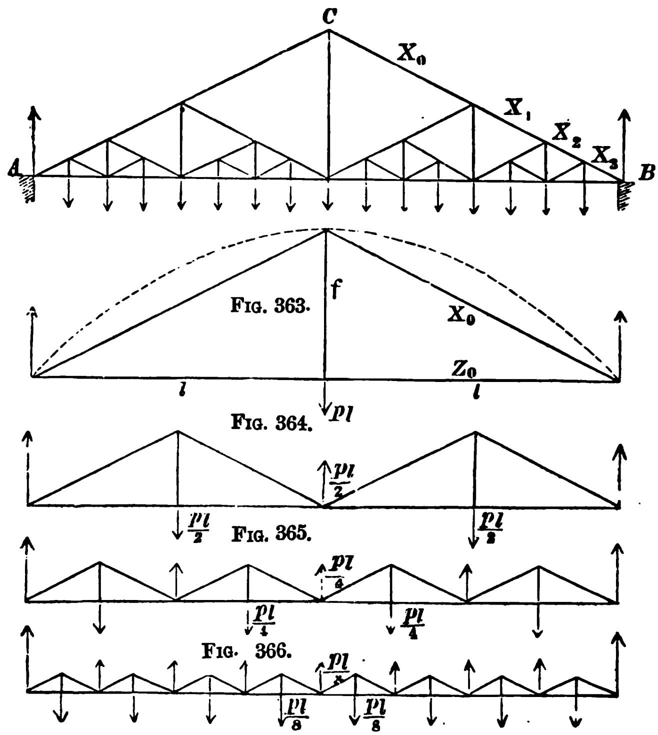
When all these intermediate bearers are placed in their proper positions with reference to the main structure, it will be found that several of the bars run side by side. These bars may, as it were, be fused together, and the stress in the

resulting bar will be the sum of the stresses in the bars of which it is composed.

In several cases it is possible and advisable to design the intermediate girders or bearers geometrically similar to the main structure. If this be done, the stresses in the resulting structure, however complicated, can be found by splitting it up into its primary forms, and often the stresses are easier determined in this manner than by employing the method of moments, as will appear from the following example :—

The truss of Fig. 362 can be considered as made up of the primary forms shown in Figs. 363, 364, 365, and 366.

Fig. 362.



Now Fig. 363 can be regarded as a parabolic girder, having only one loaded point. As already explained, the laws relating

to parabolic girders are independent of the number of loaded points. The equations of § 8, namely,

$$H = \frac{p\,l^2}{2f}, \qquad V = p\,x,$$

can therefore be used to find stresses in the bars X_0 and Z_0 by substituting $\frac{l}{2}$ for x. Thus (Fig. 363):

$$Z_0 = \frac{p}{2}\left(\frac{l}{f}\right) . \, l$$

$$-X_0 = \sqrt{H^2 + V^2} = \frac{p}{2}\left(\frac{l}{f}\right) . \, l \sqrt{1 + \left(\frac{f}{l}\right)^2}.$$

Now, since in this case the primary forms are geometrically similar, the ratio $\frac{f}{l}$ is constant for all, and the stresses corresponding to Z_0 and X_0 in the intermediate bearers of the first, second, and third order can be obtained by dividing the values found above by 2, 4, and 8 respectively.

Let the three systems of intermediate bearers be now framed into each other, and also into Fig. 363, so as to produce Fig. 362. The stresses in the different parts of the bars A C and B C will then evidently be as follows:

$$X_0 = -\frac{p\,l^2}{2f}\sqrt{1 + \left(\frac{f}{l}\right)^2}$$

$$X_1 = X_0\left(1 + \tfrac{1}{2}\right)$$

$$X_2 = X_0\left(1 + \tfrac{1}{2} + \tfrac{1}{4}\right)$$

$$X_3 = X_0\left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}\right);$$

and the stress in the horizontal bar A B is:

$$Z = Z_0\left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}\right) = \frac{p\,l^2}{2f}\left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}\right).$$

To find the stresses in the remaining bars, a similar process can be applied to Fig. 365, after Fig. 366 has been framed into it; and again, to Fig. 364, after *both* the previous figures are combined with it.

Let $2\,l = 32$ metres, $f = 6 \cdot 4$ metres, and $2\,p\,l = 32{,}000$ kilos.;

then Fig. 367 gives the various stresses, omitting, however, the system of Fig. 366. These stresses have been calculated on the supposition that the points of loading lie in the horizontal A B, but this only affects the stresses in the verticals.

It is easily seen that Fig. 368 is but a variation of the above.

Fig. 367.

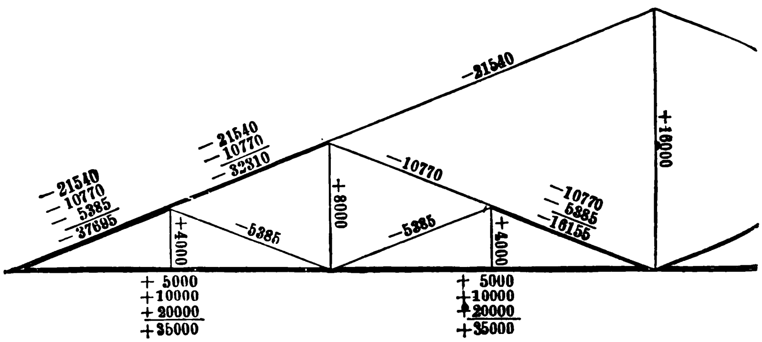
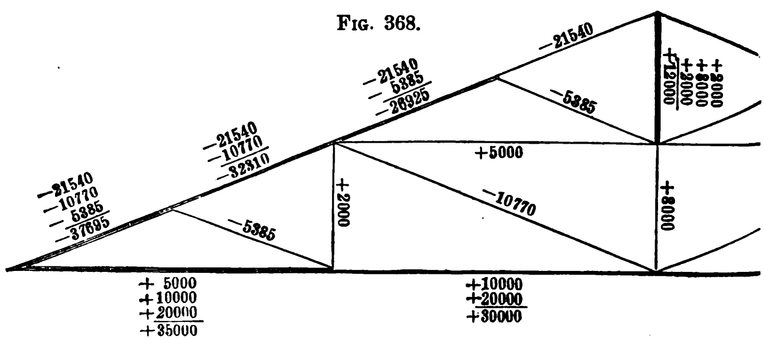


Fig. 368.

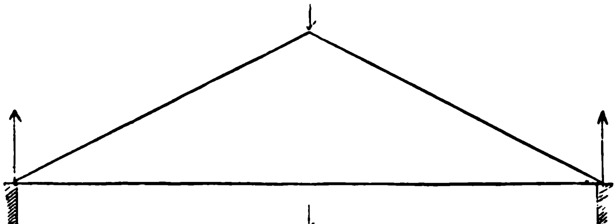


In both these examples the position, as well as the form, of the intermediate bearers corresponded to those of the main structure. Figs. 369, 370, 371, and 372, however, represent a case in which the form of the intermediate bearers is similar to that of the main structure, but in which the position is different. Apart from this difference in position, the stresses *can be* calculated as in the previous example.

Fig. 371 is the roof truss of § 4, and the stresses then obtained could have been found as follows :—

The data were $2l = 32$ metres, the height of roof $f = 6·4$ metres, and the total load $2pl = 32{,}000$ kilos. The stresses

FIG. 369.



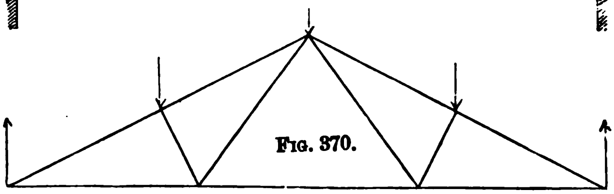


FIG. 370.

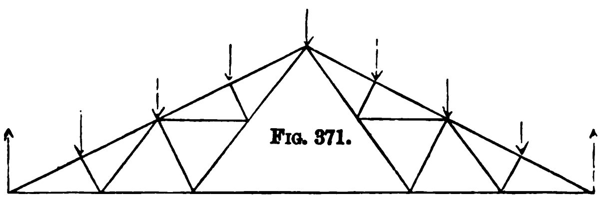


FIG. 371.

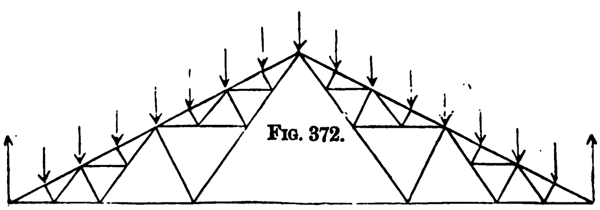


FIG. 372.

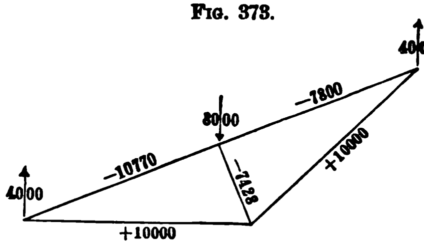
in the main triangle, Fig. 369, can be obtained as in the previous case (Fig. 363), and using the same symbols,

$$Z_0 = \frac{p\,l^2}{2f} = \frac{1000 \times 16^2}{2 \times 6·4} = 20000 \text{ kilos.}$$

$$X_0 = -\frac{p\,l^2}{2f}\sqrt{1 + \left(\frac{f}{l}\right)^2} = -20000\sqrt{1 + \left(\frac{6·4}{16}\right)^2} = -21540 \text{ kilos.}$$

Fig. 373 represents one of the intermediate bearers of the first order, and the stresses given have been calculated by the method of moments. These stresses once determined, those in the intermediate bearers of the second order can be found by dividing by two, those in the third order by dividing by four, and so on. Fig. 371 is obtained by combining the intermediate bearers of

FIG. 373.



first and second order with the main triangle, and Fig. 372 is likewise formed by the addition of the bearers of third order. The stresses in each case can be found by adding together the stresses in the separate systems where the bars coincide. Thus in Fig. 71 the stresses in the two bars meeting at the abutments are

$$20000 + 10000 + 5000 = +35000 \text{ kilos.}$$
$$-(21540 + 10770 + 5385) = -37695 \text{ kilos.}$$

And the stresses in the same bars in Fig. 372 are

$$20000 + 10000 + 5000 + 2500 = 37500 \text{ kilos.}$$
$$-(21540 + 10770 + 5385 + 2692 \cdot 5) = -40387 \cdot 5 \text{ kilos.}$$

The stresses already given in Fig. 19 can therefore be considered as made up in the manner shown in Fig. 374.

Another way of subdividing the distance between the joints of the main structure is shown in Fig. 378, and Figs. 376 and 377 show the manner in which Fig. 378 is derived from 375.

The load on this roof truss can be considered as due to a loaded horizontal beam, severed over each loaded joint, and supported at these points by vertical columns. It is easy to see that each joint receives half the load on the adjacent bays of the beam, and consequently the stresses produced in each intermediate bearer by its load can be calculated by the method of moments in the manner indicated with reference to Fig. 373.

FIG. 374.

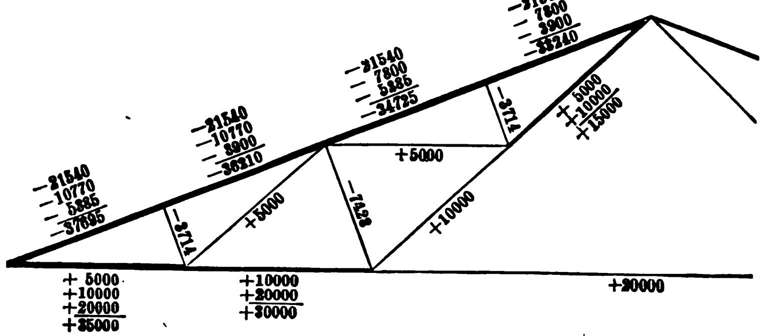


FIG. 375.

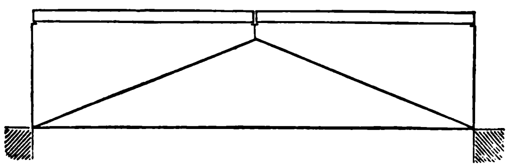


FIG. 376.

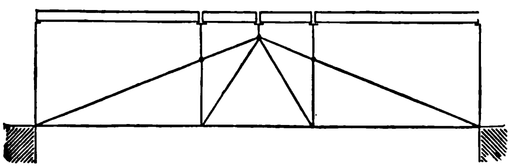


FIG. 377.

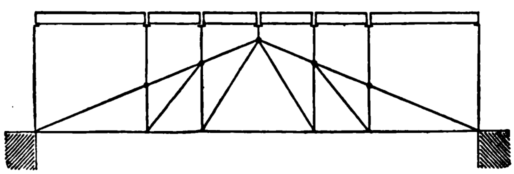
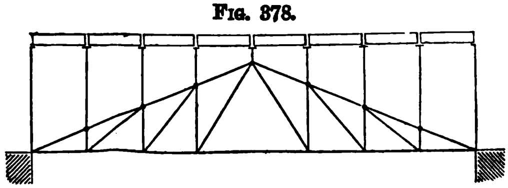


FIG. 378.



If, as in the former roofs, $2l = 32$ metres, $f = 6 \cdot 4$ metres, and $2pl = 32{,}000$ kilos., the stresses obtained are those given in Fig. 379.

The roof truss of § 3 could also be calculated in a similar manner, and the stresses given in Fig. 14 can be considered as made up as shown in Fig. 380.

FIG. 379.

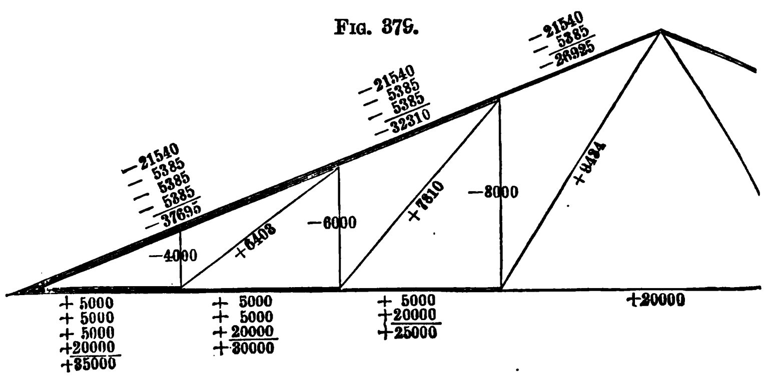
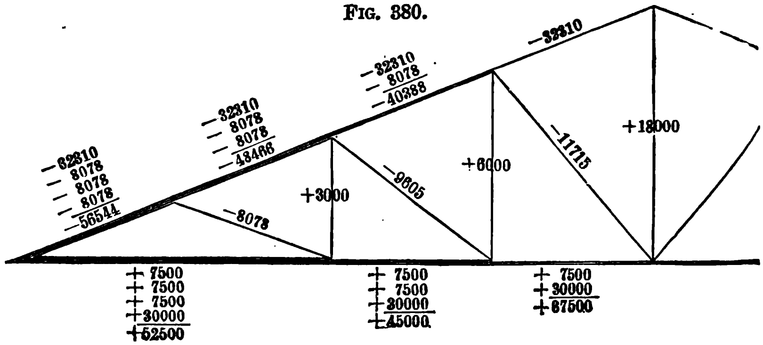


FIG. 380.



In all the above structures the application of the intermediate trusses increased the stress in the coinciding bars of the main truss because the stresses to be added together were of the same sign, and for the same reason it was also unnecessary to distinguish between the permanent and moving loads. It is, however, advantageous, if possible, so to introduce the intermediate

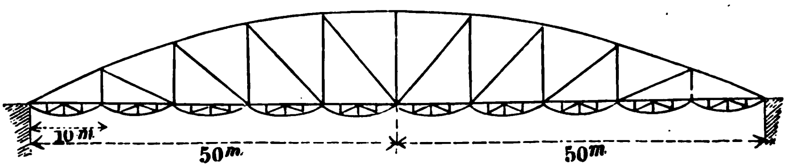
bearers that the stress in the coinciding bars may be of opposite sign, so that when fuzed together the stresses may partially neutralize each other, thus effecting a saving of material.

For instance, in a parabolic girder of 100 metres span (see § 34) the secondary trusses could be best arranged as shown in Fig. 381, when the compression booms of the secondary trusses will coincide with the tension boom of the main girder.

The tension in this latter is, with the numerical values, given in § 34.

$$H = \frac{(p+q)l^2}{2f_a} = \frac{(4+2)\,50^2}{2 \times 12 \cdot 5} = +\,600 \text{ tons.}$$

FIG. 381.



The secondary trusses are small girders of 10 metres span; they have to carry the whole of the moving load and about half of the permanent load. For example, let them be parabolic girders in which $\frac{\text{depth}}{\text{span}} = \frac{f_1}{2\lambda} = \frac{1}{5}$, then the compression in their horizontal booms is

$$h = -\frac{\left(\frac{p}{2}+q\right)\lambda^2}{2f_1} = -\frac{(2+2)\,5^2}{2 \times 2} = -\,25 \text{ tons.}$$

Therefore when the bridge is fully loaded the tension in the main lower boom would be

$$H + h = 600 - 25 = 575 \text{ tons,}$$

instead of 600 tons.

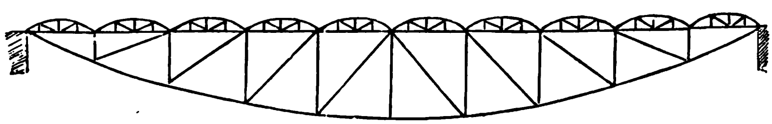
The cross-section of the main booms could not, however, be diminished, for if only one bay were unloaded the stress in the part of the lower boom belonging to that bay would scarcely be reduced. But the whole of the material for the upper booms

of the secondary trusses is saved. Since the quantity of material in their curved booms is very little in excess of that required for the upper booms, the material in the whole of the secondary trusses will be to that in the main girder as the horizontal stress in the secondary trusses is to twice the horizontal stress in the main girder, or in the ratio,

$$\frac{25}{2 \times 600} = \frac{1}{48}.$$

If the main girder is reversed as in Fig. 382, the secondary trusses will also have to be reversed to obtain the same advantage.

FIG. 382.



It is hardly necessary to remark that the parabolic form has been chosen for the secondary girders only as an illustration, and that they could be constructed as lattice girders, or, if the void spaces between the bars become too small, as plate girders. In this latter case the secondary girders take the form of strengthening ribs to prevent the horizontal boom from bending, and it follows that the best position for this rib is above the boom when the line of railway is on a level with the top boom, and below the boom when the line is level with the lower boom.

In lattice girders with crossed diagonals an intermediate point of support can be obtained by making the intersection of the diagonals a point of loading, the load being conveyed to it by means of a vertical. (See Fig. 384.)

By introducing triangular intermediate girders, the number of points of support can be increased exactly in a similar manner to that adopted for the roof trusses. Thus from the fundamental form of Fig. 383, the girders shown in Fig. 384, 385, and 386 are derived.

As to the manner of calculating the stresses in these girders. The stresses in Fig. 384 would first be found by the method of moments, and for the stresses in the booms the intersection

of the diagonals would be chosen as the turning-points. The stress in any diagonal can be found by remembering that the vertical component of the required stress is *in this case* half the vertical force at any section (in other words, half the shearing force). The stresses in Fig. 385 can then be obtained by following the method already explained for roof trusses.

FIG. 383.

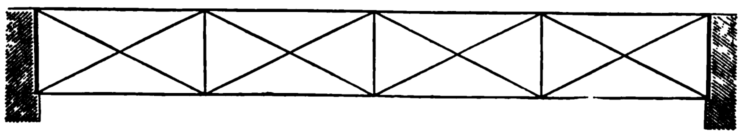


FIG. 384.

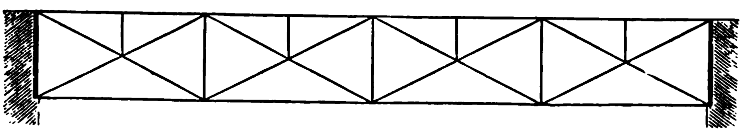


FIG. 385.

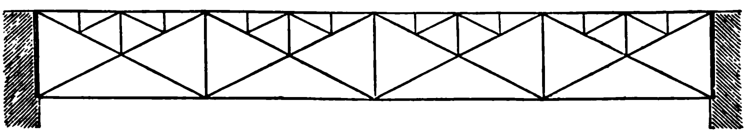
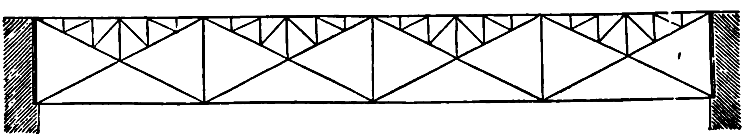


FIG. 386.



It must be observed that it makes a difference in the calculated stresses whether the girder of Fig. 385 be considered to have seventeen loaded joints (which it really has) or only the nine belonging to the main structure (Fig. 384). (In the last case the weight of the intermediate trusses is supposed to be transmitted to the joints of the main girder.)

Evidently the assumption that, as the moving load proceeds, one joint is fully loaded before the next receives any load at all, is, as pointed out at the end of § 12, not strictly true, but it is also evident that the greater the number of loaded joints—that is, the nearer they are to each other—the less will be the result-

ing error. It follows that the stresses given in Fig. 387 obtained under the supposition that the girder has only nine loaded joints, will differ slightly from those that would be obtained if it were considered to have seventeen loaded joints, the latter being the more accurate values. It will also be observed, when it is assumed that there are nine loaded joints and the moving load has arrived over the centre of one of the intermediate girders, that the next following joint of the main girder has already received a part of its load, for it acts as a point of support to the intermediate girder; and when the moving load has arrived at the end of one of the intermediate girders, the joint of the main girder at this point has not yet received its full load, for it acts as a point of support to the next intermediate girder which is as yet unloaded.

For the sake of comparison, the dimensions, &c., in Figs. 387 and 388 are the same as those of the girder calculated in § 10, namely, depth = 2 metres, span = 16 metres, total load on the girder = 48,000 kilos. (consisting of, permanent load = 8000 kilos., moving load = 40,000 kilos.).

The stresses calculated from these data for the girders of Figs. 384 and 385 are given in Figs. 387 and 388.

FIG. 387.

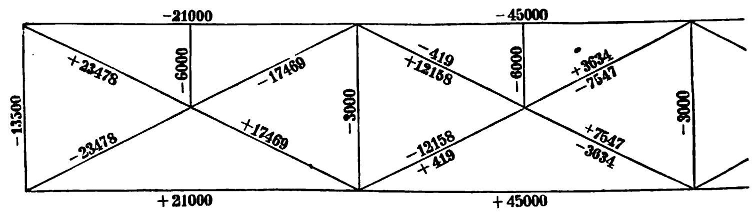
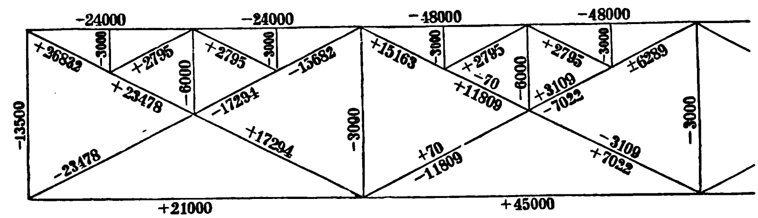


FIG. 388.



By comparing these two girders with those of § 14, it will be seen that the number of points of support in a simple

girder such as that shown in Fig. 383 can be increased in two different ways, viz. either by increasing the number of triangulations or else by introducing intermediate trusses as explained above.

The first method would evidently be the best if the sectional area required to resist compression were always proportional to the stress; for then it would be possible to increase the number of points of support indefinitely without adding to the quantity of material, whereas in the second method every new diagonal and vertical requires extra material. But this advantage is only apparent, because with multiple latticing the compression braces become very thin and are therefore liable to bend or buckle, and hence require much larger cross-sections in proportion.

The advantages and disadvantages of each system must however be considered specially in each case, for this is a point which cannot be decided generally.

NOTE.—In practice it is usual to rivet the braces together where they intersect. This, in one respect, is not right, for the braces are thereby impeded from acting independently, but the great advantage is obtained that the tendency to buckle of the compression braces is greatly reduced, and the objection to multiple lattice girders, mentioned above, is avoided.

THIRTEENTH CHAPTER.

§ 44.—On the Deflection of Loaded Structures.

It has been found by experiment that the amount of alteration in the length of a bar is proportional to the stress, so long as the stress is within the limits of elasticity, and this whether the stress be compression or tension.

Thus if δ is the alteration of length in one unit of length due to a stress S per unit of area,

$$\delta \propto S;$$

or

$$\delta = \frac{S}{E}; \qquad [I.]$$

where E is a constant quantity.

E is called the *modulus of elasticity*, and from equation I. it is evident that $\frac{1}{E}$ is the elongation or shortening in one unit of length produced by the unit stress per unit of area. Another definition of E can be deduced from equation I., namely, that it is the stress per unit of area that will lengthen a bar to double its original length.

Its value for wrought iron is about 20,000 when expressed in millimetres and kilogrammes (equivalent to 28,450,000 when the English inch and the lb. avoirdupois are the units). Thus every millimetre of the length of a wrought-iron rod, whatever be the section, increases its length by $\frac{1}{20000}$ millimetre when a tension of 1 kilo. per square millimetre is applied to it. If, however, the rod were subject to a tension of 6 kilos. per square millimetre, each millimetre would increase its length by $\frac{6}{20000}$ or $\frac{1}{3333}$ millimetre.

The actual increase of length is obtained by multiplying the original length l by δ, thus:

$$\lambda = l\delta. \qquad [II.]$$

A wrought-iron rod, therefore, of 10 metres length, subject to a tension of 6 kilos. per square millimetre, increases its length by

$$\lambda = 10000 \times \frac{3}{10000} = 3 \text{ millimetres.}$$

Negative stress or compression produces negative elongation or shortening in the same proportion as above; therefore equations I. and II. can be used for compression as well as for tension.

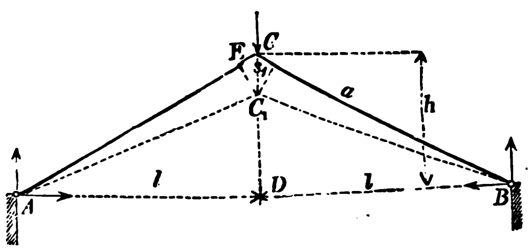
If the stresses in a loaded structure are known, it is a purely geometrical problem to find the change of form and the consequent deflection; for the structure can be imagined taken to pieces and then remade, using the altered lengths of the several bars.

A few examples will now be given to show how the application of the above laws can be transformed into a geometrical problem. It is also of use to know the deflection of various simple structures under a known load, for this will enable the proportion to be found in which the load on a complex structure subdivides itself between its component simple structures whence the stresses in these latter can be deduced.

It will be assumed that in all cases the cross-section of any bar is proportional to the stress in it, so that the elongation or shortening of each part of the structure will be δ for each unit of length.

Thus in Fig. 389, owing to the shortening $a\,\delta$ of the two bars A C and C B due to a load at C, the point C is lowered to C_1.

FIG. 389.



The position of C_1 can be found by describing arcs from A and B with radii equal to $a - a\,\delta$. If $E\,C = a\,\delta$, $E\,C_1$ can be drawn at

right angles to A C without appreciable error, because $a\,\delta$ is small, and the same is true for the other side.

The triangles C E C_1 and C D A being similar,

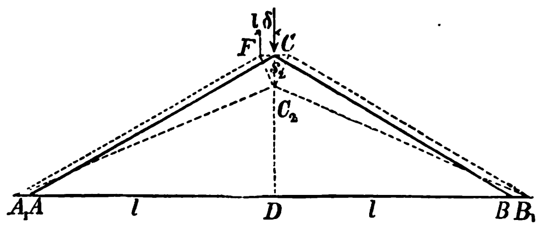
$$\frac{C\,C_1}{C\,E} = \frac{s_1}{a\,\delta} = \frac{a}{h};$$

or

$$s_1 = \delta \frac{a^2}{h}. \qquad [1]$$

When the abutments are relieved from thrust by means of a tie rod A B, the point C is further lowered by an amount S_2 due to the increase of length $l\,\delta$ of A D. Suppose that at first A C and B C do not alter their length, then the position of C_2 can be found by describing arcs from A_1 and B_1 with radii a (Fig. 390).

FIG. 390.



From the similarity of the triangles C F C_2 and C D A,

$$\frac{C\,C_2}{CF} = \frac{s_2}{l\,\delta} = \frac{l}{h};$$

or

$$s_2 = \delta \frac{l_2}{h}. \qquad [2]$$

The total depression of the point C is therefore

$$s = s_1 + s_2 = \delta \left(\frac{a^2 + l^2}{h}\right). \qquad [3]$$

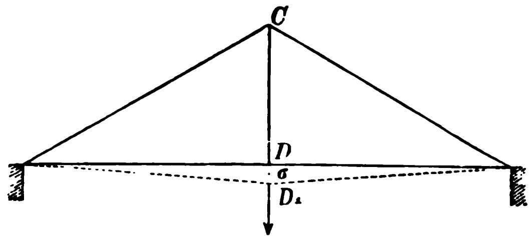
If the load, instead of being applied at C, is hung at D (Fig. 391) and transmitted to C by means of the tension rod C D, the point D will be lowered by an amount s', which is the sum

of the depression of the point C and of the increase in length σ of the C D, thus

$$s' = s_1 + s_2 + \sigma = \delta\left(\frac{a^2 + l^2 + h^2}{h}\right) = 2\,\delta\,\frac{a^2}{h}. \qquad [4]$$

The above equations are also true for the reversed position of the structure.

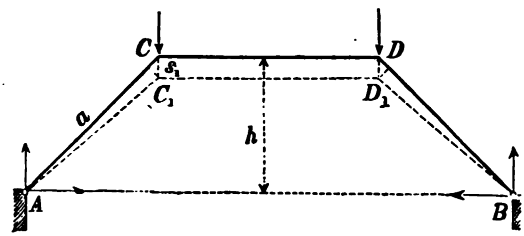
FIG. 391.



In a similar manner the depressions of the points C and D in Fig. 392 can be determined by first finding the part due to the shortening of A C and D B:

$$s_1 = \delta\,\frac{a^2}{h},$$

FIG. 392.



and then that due to the shortening of C D (Fig. 393):

$$s_2 = \delta\,\frac{b\,c}{h}.$$

The total depression is therefore

$$s = s_1 + s_2 = \delta\left(\frac{a^2 + b\,c}{h}\right). \qquad [5]$$

When the points A and B are connected together by a tension rod (Fig. 394) a further depression of

$$\sigma_1 = \delta \frac{(b+c)c}{h}$$

is produced, and in this case C and D will be depressed by an amount

$$s = s_1 + s_2 + \sigma_1 = \delta \left(\frac{a^2 + bc + (b+c)c}{h}\right). \qquad [6]$$

FIG. 393.

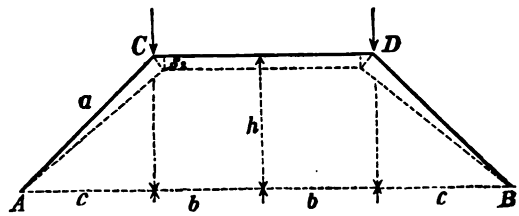


FIG. 394.

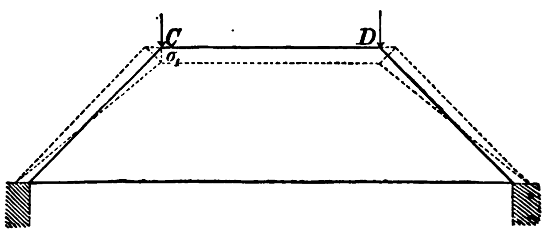
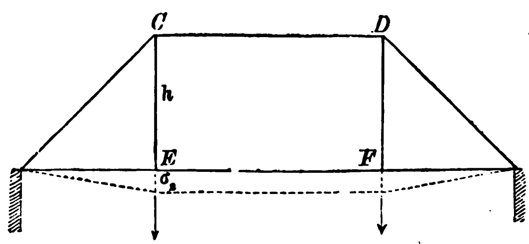


FIG. 395.



Lastly, when the loads are hung at E and F, Fig. 395, *these points* are lowered by an amount equal to σ_2, due to the

lengthening of the tie rods, added to the depression of the point C or D found above. But $\sigma_2 = h\,\delta$. Consequently

$$s = s_1 + s_2 + \sigma_1 + \sigma_2 = \frac{\delta}{h}\left[a^2 + b\,c + (b + c)\,c + h^2\right]$$

$$= 2\,\frac{\delta}{h}\,(a^2 + b\,c). \qquad [7]$$

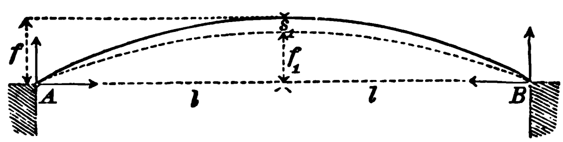
This method of investigation can be extended to the case when there are three or more points of loading.

§ 45.—Deflection of Parabolic Arches and Girders.

The effect of the load on the inverted parabolic chain in Fig. 396 is to shorten the length of arc, and thus reduce its height from f to f_1. The deflection s_1 at the crown can therefore be found from the equation (Fig. 396)

$$s_1 = f - f_1. \qquad [8]$$

Fig. 396.



When the ratio $\frac{f}{2l}$ of the height of the arc to the span is small, it can be shown that the length of the arc of the original parabola is given by the equation

$$S = 2l\left(1 + \tfrac{2}{3}\frac{f^2}{l^2}\right);^*$$

* The equation to the parabola is

$$\frac{y}{f} = \frac{x^2}{l^2}.$$

Differentiating with respect to x,

$$\frac{d\,y}{d\,x} = 2\,f\,\frac{x}{l^2}.$$

[But

Consequently the length of arc of the compressed parabola is

$$S_1 = S(1-\delta) = 2l\left(1+\tfrac{2}{3}\frac{f_1^2}{l^2}\right),$$

or substituting for S its value

$$2l\left(1+\tfrac{2}{3}\frac{f^2}{l^2}\right)(1-\delta) = 2l\left(1+\tfrac{2}{3}\frac{f_1^2}{l_2}\right).$$

and solving this equation for f_1,

$$f_1 = f\sqrt{1-\delta\left(1+\tfrac{3}{2}\frac{l^2}{f^2}\right)}$$

$$= f\left\{1-\frac{\delta}{2}\left(1+\tfrac{3}{2}\frac{l^2}{f^2}\right)-\frac{\delta^2}{8}\left(1+\tfrac{3}{2}\frac{l^2}{f^2}\right)^2\ldots\right.$$

The error entailed by leaving out the expression $\frac{\delta^2}{8}\left(1+\frac{3}{2}\frac{l^2}{f^2}\right)^2$ and those that follow it is small, and hence, although $\frac{l}{f}$ is assumed large,

$$f_1 = f\left[1-\frac{\delta}{2}\left(1+\tfrac{3}{2}\frac{l^2}{f^2}\right)\right]$$

approximately.

Substituting this value of f_1 in equation 8,

$$s_1 = \frac{f\delta}{2}\left(1+\tfrac{3}{2}\frac{l^2}{f^2}\right) = \tfrac{3}{4}\frac{\delta l^2}{f}\left(1+\tfrac{2}{3}\frac{f^2}{l^2}\right). \qquad [9]$$

But the differential of the arc S is given by

$$dS = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= dx\sqrt{1+\frac{4f^2x^2}{l^4}}$$

$$= dx\left(1+\frac{2f^2x^2}{l^4}-\frac{2f^4x^4}{l^8}+\ldots\right),$$

or approximately, if $\frac{f}{2l}$ is small,

$$dS = dx\left(1+\frac{2f^2}{l^4}x^2\right)$$

Integrating between the limits $-l$ and $+l$,

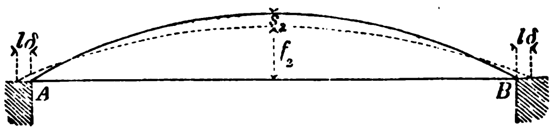
$$S = \int_{-l}^{+l}\left(1+\frac{2f^2}{l^4}x^2\right)dx = 2l\left(1+\tfrac{2}{3}\frac{f^2}{l^2}\right).$$

For example, let $l = 20{,}000$ millimetres, $f = 5000$ millimetres, $\delta = \frac{6}{20000}$, then $s_1 = 18\,(1 + \frac{1}{24}) = 18{\cdot}75$ millimetres, and this is the amount the crown of the parabolic arch of § 22 sinks when the load produces a compression of 6 kilos. per square millimetre in the bow.

Evidently equation 9 is applicable to a suspension bridge, and in this case δ will be the increase of length per unit of length.

If the points A and B are connected together by a tie, the crown will sink by a further amount s_2, due to the extension of this tie. The structure then becomes a girder, since the reactions at the abutments are vertical.

FIG. 397.



s_2 can evidently be found by assuming that the tie alone extends by an amount $2\,l\,\delta$, and that the length of the bow remains unchanged. Therefore equating the length of arc of the original parabola to that of a parabola having a height of arc f_2 and a span $2\,l\,(1 + \delta)$,

$$2\,l\left(1 + \tfrac{2}{3}\frac{f^2}{l^2}\right) = 2\,l\,(1 + \delta)\left(1 + \tfrac{2}{3}\frac{f_2^{\,2}}{l^2\,(1 + \delta)^2}\right);$$

and solving this equation approximately for f_2,

$$f_2 = f\left[1 + \frac{\delta}{2}\left(1 - \tfrac{3}{2}\frac{l^2}{f^2}\right)\right].$$

The depression of the crown due to the extension of the tie is therefore

$$s_2 = f - f_2 = \frac{f\,\delta}{2}\left(\tfrac{3}{2}\frac{l^2}{f^2} - 1\right) = \tfrac{3}{4}\frac{\delta\,l^2}{f}\left(1 - \tfrac{2}{3}\frac{f^2}{l^2}\right). \qquad [10]$$

The total depression of the crown when the bow is compressed will evidently be

$$s = s_1 + s_2 = \tfrac{3}{2}\,\delta\,\frac{l^2}{f} \qquad [11]$$

If the load is communicated from the lower boom to the upper boom by means of vertical rods, the extension of these rods will lower the centre of the tie by a further amount δf.

But since the above results were obtained on the supposition that the ratio $\frac{f}{l}$ was small, δf can be neglected in comparison to s, more especially as in actual girders the greatest tension in the verticals occurs with a partial load.

Equation 11 is also true for a parabolic girder having the bow underneath.

Example.—In the parabolic girder calculated in § 6, the span was 16 metres, and the height of arc 2 metres; let $\delta = \frac{6}{20000}$, then the deflection at the crown is

$$s = \frac{3}{2} \cdot \frac{6}{20000} \cdot \frac{8000^2}{2000} = 14 \cdot 4 \text{ millimetres.}$$

§ 46.—Deflection of Braced Girders with Parallel Booms.

In girders with parallel booms the deflection s (in the centre) is composed of two parts, one entirely due to the booms and the other to the braces.

To find the first part, s_1, the centre line of the girder can be considered as bent into the arc of a circle of radius ρ (Fig. 398). Since s_1 is small, the length of arc differs but by a very small quantity from the chord; the length of the chord is therefore $2\,l$, and the equation to the circle is

$$l^2 = 2\,\rho\,s_1 - s_1^2;$$

but s_1^2 is small in comparison to $2\,\rho\,s_1$. Hence,

$$s_1 = \frac{l^2}{2\,\rho}. \qquad [12]$$

Further, the lengths of arc of the outer circle (formed by the lower boom) and the centre circle are as their radii, or

$$\frac{\rho + \frac{f}{2}}{\rho} = \frac{2\,l\,(1 + \delta)}{2\,l};$$

whence

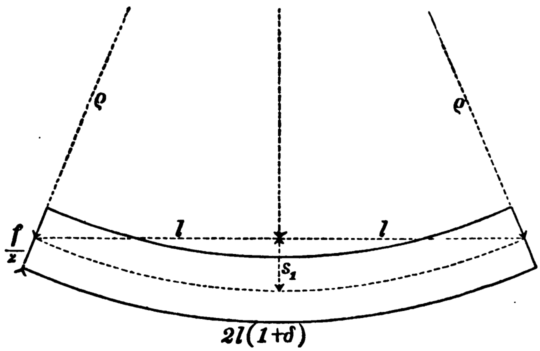
$$\rho = \frac{f}{2\,\delta}. \qquad [13]$$

Substituting this value of ρ in equation 12,

$$s_1 = \frac{l^2\,\delta}{f}. \qquad [14]$$

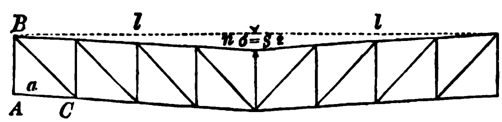
The deflection s_2, due to the braces alone, can be obtained by considering that the booms remain unaltered. and then finding

FIG. 398.



the change of form produced in each of the right-angled triangles formed by a vertical, a diagonal, and a horizontal bar, by the lengthening of the diagonal and the shortening of the vertical (Fig. 399).

FIG. 399.



The point C of the triangle A B C is lowered by an amount σ in consequence of this change of form; and σ can be considered as composed of two parts. The first, ϵ, is due to the lengthening of the diagonal, and the second, λ, to the shortening of the vertical. The extension of the diagonal is $c\,\delta$, and ϵ can therefore be obtained from the equation (Fig. 400)

$$\frac{\epsilon}{c\,\delta} = \frac{c}{f}.$$

The shortening of the vertical is λ, and evidently

$$\lambda = f\delta.$$

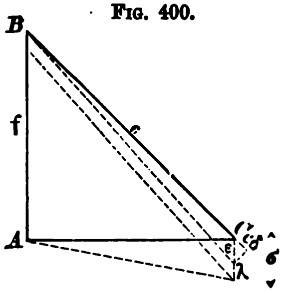
Hence,

$$\sigma = \epsilon + \lambda = \delta f\left(\frac{c^2}{f^2} + 1\right);$$

or, since $c^2 = f^2 + a^2$,

$$\sigma = a\delta\frac{f}{a}\left(\frac{a^2}{f^2} + 2\right).$$

FIG. 400.



If n is the number of bays between the abutments and the centre, the deflection s_2 will be n times σ. Hence

$$s_2 = n\sigma = na \,.\, \delta\frac{f}{a}\left(\frac{a^2}{f^2} + 2\right),$$

or, since $n\,a = l$,

$$s_2 = l\delta\frac{f}{a}\left(\frac{a^2}{f^2} + 2\right). \qquad [15]$$

By adding together the values of s_1 and s_2 (from equations 14 and 15), the total deflection at the centre of the girder shown in Fig. 399 is found to be

$$s = l\delta\left[\frac{l}{f} + \frac{f}{a}\left(\frac{a^2}{f^2} + 2\right)\right]. \qquad [16]$$

If the bays are square,

$$\frac{f}{a} = 1 \quad \text{and} \quad s = l\delta\left(\frac{l}{f} + 3\right);$$

and if at the same time there are eight bays,

$$\frac{l}{f} = 4 \quad \text{and} \quad s = 7l\delta.$$

Example.—The deflection at the centre of the girder calculated in § 10 can be found by means of this formula. The span is 16 metres, and assuming that $\delta = \frac{6}{10000}$ millimetres,

$$s = 7 \times 8000 \times \frac{6}{10000} = 16 \cdot 8 \text{ millimetres.}$$

If the braces were constructed much stronger than necessary, the alteration in their length, and consequently the *deflection* due to them, might be neglected. The deflection s

(equation 14), due to the booms alone, would then be the deflection at the centre of the girder, assuming that the deflection curve is the arc of a circle. These conditions are approximately fulfilled in a plate-web girder; the deflection of such a girder can therefore be found approximately from equation 14. This equation also *contains* the general law of the deflection of a beam of equal depth and symmetrical cross-section throughout; for the deflection at the centre can never be greater than when the bending or curvature at every point is a maximum—that is when the deflection curve is a circle. This is actually the case in a beam of uniform strength; that is, when the form of the beam is so proportioned to the load that the greatest stress in every cross-section is constant.

FOURTEENTH CHAPTER.

§ 47.—Theory of Composite Structures.

The two following questions can be answered by means of the equations obtained in the preceding chapter, or by means of others similar to them:—

1. In a structure composed of two distinct systems connected together, what strength and stiffness should each system have in order that they may work *evenly* together?

2. What is the proportion of the total load carried by each system?

As the common points of loading of the two simple systems gradually sink owing to the increase of the load, the stresses in each will augment, and alterations in the lengths of the various bars, &c., will take place. So soon, however, as the limit of elasticity in one of the systems is reached, the load could not *safely* be increased, however remote the stresses in the other system may be from their limit of elasticity. This second system might therefore be made of weaker and less elastic material without diminishing the *safe* resistance of the whole structure, or, better still, the quantity of material in it might be reduced, so that in both systems the limit of elasticity would be reached at the same time. All the material thus saved is not only unnecessary to the structure, but is positively harmful, in that it adds to the dead load. Such combined structures should therefore, if possible, be so designed that the simple systems composing them are *equally stiff*.

The proportion of the load carried by each system can, however, be found, whether this condition be complied with or not, by equating the deflection of each system at the points where they are connected together, and these are also the points where the loads are transmitted from one system to the other.

An equation is thus obtained, giving $\frac{\delta_1}{\delta_2}$, the ratio of the alterations of length per unit of length; and from this it is easy to obtain the ratio $\frac{Q_1}{Q_2}$, giving the proportion of the load carried by each system.

It appears from the previous investigations, that the general equation giving the deflection of a loaded point can be written

$$s = A\,\delta,$$

where A is a constant, depending on the form and dimensions of the structure (the value of this constant can be obtained for the various cases considered in the last chapter, from equations 1 to 16). The deflection of each simple system at some point where they are connected together will therefore be $A_1\,\delta_1$ and $A_2\,\delta_2$ respectively. Hence,

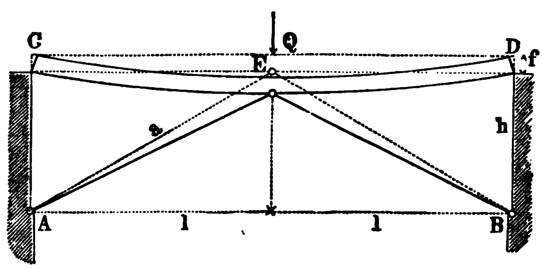
$$A_1\,\delta_1 = A_2\,\delta_2,$$

or

$$\frac{\delta_1}{\delta_2} = \frac{A_2}{A_1}. \qquad \text{[III.]}$$

Now the loads producing the alteration in length δ_1 and δ_2 per unit of length in each of the simple systems respectively can be found by the methods already explained; and since the total load on the structure is known, the part carried by each system can easily be found.

FIG. 401.



For example, let two simple systems, one like Fig. 389 and the other like Fig. 398, be combined together as shown in Fig. 401. If both systems are made of the same material,

the condition that they may reach the elastic limit together can be found by putting $\delta_1 = \delta_2$. Hence, from equation III.,

$$A_1 = A_2,$$

or from equations 1 and 14,

$$\frac{a^2}{h} = \frac{l^2}{f};$$

whence

$$\frac{f}{h} = \frac{l^2}{a^2}. \qquad [17]$$

[The value thus found for $\frac{f}{h}$ may be termed "the economical ratio of the depths" (with regard only to the quantity of material, not necessarily as regards the cost).]

If the elastic limit is not reached simultaneously, or if each system is made of a different material, it appears from equation III. that

$$\frac{\delta_1 a^2}{h} = \frac{\delta_2 l^2}{f} \quad \text{or} \quad \frac{\delta_1}{\delta_2} = \frac{\frac{l^2}{f}}{\frac{a^2}{h}}. \qquad [18]$$

The economical ratio of the depths can be found in this case by putting for δ_1 and δ_2 their values at the limit of elasticity, for the material of which the corresponding system is made.

From equation 18 the ratio $\frac{Q_1}{Q_2}$ can be found, as follows: Let the load Q_1 acting on the simple system formed by the rods A E and B E, produce a stress S_1 per unit of area in each of the rods, and let F_1 be the sectional area of each rod; then

$$S_1 = \frac{Q_1 a}{2 F_1 h};$$

therefore, the shortening per unit of length in these two rods is

$$\delta_1 = \frac{S_1}{E_1} = \frac{Q_1 a}{2 F_1 h E_1}. \qquad [19]$$

Again, if F_2 is the sectional area of either of the booms of the girder C D at the point E, the stress S_2 per unit of area produced by the load Q_2 is

$$S_2 = \frac{Q_2 l}{2 F_2 f},$$

and the consequent alteration of length per unit of length in this girder is

$$\delta_2 = \frac{S_2}{E_2} = \frac{Q_2 l}{2 F_2 f E_2}. \qquad [20]$$

By substituting these values of δ_1 and δ_2 in equation 18, the following equation is obtained:—

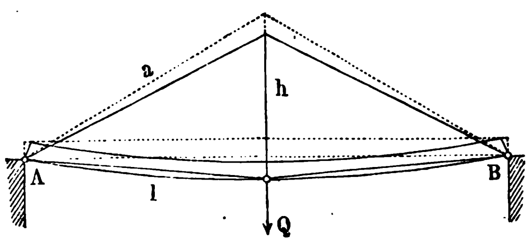
$$\frac{Q_1}{Q_2} = \frac{l^3}{a^3} \cdot \frac{h^2}{f^2} \cdot \frac{F_1}{F_2} \cdot \frac{E_1}{E_2}, \qquad [21]$$

from which, since $Q_1 + Q_2 = Q$ is known, the distribution of the load can be ascertained. If the structure is made of the same material throughout and complies with the condition expressed by equation 17, equation 21 becomes

$$\frac{Q_1}{Q_2} = \frac{a}{l} \cdot \frac{F_1}{F_2}. \qquad [22]$$

In the structure shown in Fig. 402, composed of Figs. 391 and 398, the condition that the limit of elasticity may be

FIG. 402.



reached in each system at the same time when the material is the same in both, is (equations 4 and 14),

$$\frac{f}{h} = \frac{l^2}{2a^2}. \qquad [23]$$

The general equation for the distribution of the load can be obtained in the manner indicated in the previous example, and is

$$\frac{Q_1}{Q_2} = \frac{l^3 h^2}{2 a^3 f^2} \cdot \frac{F_1}{F_2} \cdot \frac{E_1}{E_2}, \qquad [24]$$

which, combined with equation 23, gives for the special case expressed by that equation

$$\frac{Q_1}{Q_2} = 2\frac{a}{l} \cdot \frac{F_1}{F_2} = \frac{h l F_1}{f a F_2}. \qquad [25]$$

The last three equations are also true for the reverse arrangement shown in Fig. 403.

The three corresponding equations for the structure given in Fig. 404 can be obtained in a similar manner, by means of

FIG. 403.

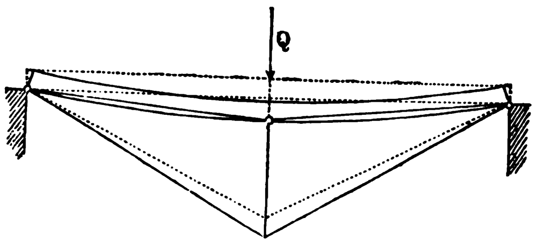
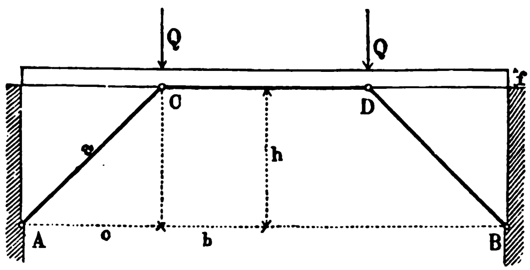


FIG. 404.



equations 5 and 14; but in this case the deflection of the girder at the points C and D must be considered, and it is equal to the deflection at the centre of the girder multiplied by $1 - \frac{b^2}{l^2}$, for the deflection varies as the square of the distance from the centre. Hence the condition that each system may reach the elastic limit at the same time (supposing both to be made of the same material) is

$$\frac{f}{h} = \frac{l^2 - b^2}{a^2 + bc}. \qquad [26]$$

The general equation for the distribution of the load is

$$\frac{Q_1}{Q_2} = \frac{h^2\,(l^2 - b^2)\,c}{f^2\,a\,(a^2 + bc)} \cdot \frac{F_1}{F_2} \cdot \frac{E_1}{E_2}, \qquad [27]$$

which, when combined with equation 26, gives for the special case expressed by that equation

$$\frac{Q_1}{Q_2} = \frac{c\,h}{a\,f} \cdot \frac{F_1}{F_2}. \qquad [28]$$

Exactly in a similar manner, the following equations are obtained for Figs. 405 and 406, by means of equations 7 and 14.

$$\frac{f}{h} = \frac{l^2 - b^2}{2\,(a^2 + b\,c)} \qquad [29]$$

$$\frac{Q_1}{Q_2} = \frac{h^2 \;\; (l^2 - b^2)\,c}{2\,f^2\,a\,(a^2 + b\,c)} \cdot \frac{F_1}{F_2} \cdot \frac{E_1}{E_2} \qquad [30]$$

$$\frac{Q_1}{Q_2} = \frac{c\,h}{a\,f} \cdot \frac{F_1}{F_2}. \qquad [31]$$

FIG. 405.

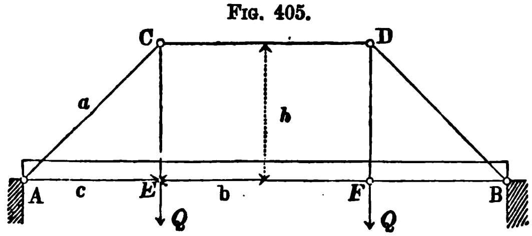
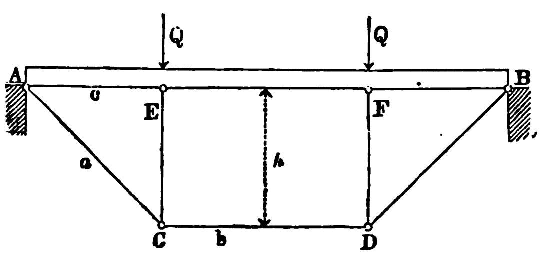


FIG. 406.

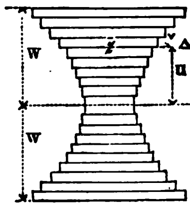


In all the preceding cases it was assumed that the alteration in length of each unit of length was the same throughout the same system, or, in other words, that the cross-section of each bar was proportional to the stress in it.

If a structure is composed of several simple systems, each having a different degree of stiffness, the total load on the structure will be distributed unequally between them, whatever the load may be, and those which owing to their want of stiffness reach their elastic limit last, evidently contain too much material.

A solid beam can be considered as composed of an infinite number of lattice or plate girders of unequal depths (Fig. 407), for imagine the beam divided into horizontal layers, then any two layers equidistant from the central line will represent the booms, and the material between them the braces or the plate web.

FIG. 407.



These imaginary girders are not equally stiff, and therefore there is a waste of material in a solid beam.

Now, all these imaginary girders have the same deflection, and from equation 14 it appears that if the deflection remains constant the alteration of length of the booms, and consequently the stress in them, is proportional to the height of the girder. The stress in each layer is therefore proportional to its distance from the centre, so that if the outer layer, at a distance w from the centre, is subject to a stress S per unit of area, the stress in a layer whose distance is u from the centre will be

$$s = \frac{u}{w} S.$$

If the stress S is known at any cross-section of the beam, the stress in any layer in the same cross-section can be obtained from this equation, and the stress in the layer symmetrically placed on the opposite side of the centre has the same numerical value, but with a contrary sign.

The total stress in such a layer is, if Δ be its thickness and z its breadth,

$$\frac{u}{w} S z \Delta.$$

If such a beam be bent by the forces K_1, K_2 .... acting at right angles to its axis, the maximum stress S at any section through P can be found by the

FIG. 408.

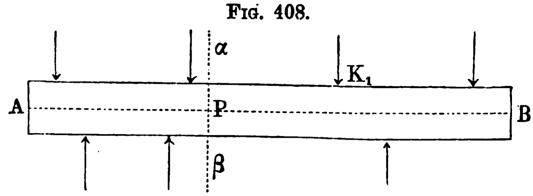
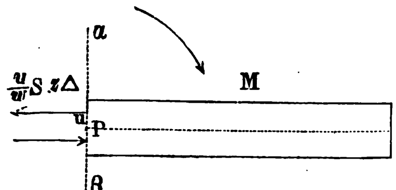


FIG. 409.



method of moments. Take a section $\alpha \beta$ (Figs. 408 and 409), and, to maintain equilibrium, apply to each layer the total stress in it. In order that there may be no rotation about P, the algebraical sum of the moments, M, of all the

exterior forces acting on the part B $\alpha\beta$ about P must be equal to the sum of the moments of all the stresses about the same point. Therefore

$$M = \Sigma\left(\frac{u}{w} S z \Delta u\right),$$

where Σ is a symbol indicating that the sum has been taken of all the separate moments due to the stress in the various layers. The factor $\frac{S}{w}$ is common to all these moments, and can therefore be placed outside the symbol, or

$$M = \frac{S}{w} \Sigma (z u^2 \Delta).$$

Now $\Sigma (z u^2 \Delta)$ is the "moment of inertia" * of the cross-section about the axis round which moments were taken, that is, about a horizontal axis in the plane of the section and passing through its centre. The moment of inertia is usually denoted by I. Hence

$$M = \frac{S}{w} I,$$

or

$$S = \frac{w}{I} M.$$

The value of I for any form of cross-section can be obtained by dividing up the area into very small parts (or elements), and multiplying the area of each element by the square of its distance from the horizontal axis through the centre of gravity, and then adding together the products thus obtained.†

§ 48.—Trussed Beams without Diagonals.

It was shown, § 8, that the diagonals of parabolic trusses only come into play with a partial load, and also that they then become an indispensable part of the structure, that is, if the various bars are connected by single bolts, and are thus only capable of taking up stress in the direction of their length. If, however, the diagonals are omitted, one of the

* The moment of inertia is a term belonging to the dynamics of a rigid body, and the following is the definition:—

If the mass of every particle of a body be multiplied by the square of its distance from a straight line, the sum of the products so formed is called the "moment of inertia" of the system about that line, which is also called the *axis*.

In the present case, it will be observed, the mass of every particle or element has been taken as unity; also from the symmetry of Fig. 407 it is apparent that the axis is horizontal and passes through the centre of gravity of the section. As will be seen in the sequel, the resistance to flexure at any cross-section of a body always depends on the moment of inertia of that cross-section about a horizontal axis in the plane of and passing through the centre of gravity of the cross-section.—Trans.

† The continuation of this subject will be found in the Fifteenth Chapter.

booms must be stiffened to enable the structure to resist the deformation that would otherwise take place with a partial load.

The stresses will, however, no longer be the same as those in a simple parabolic girder, and the structure must, in fact, be regarded as a combination of two systems, and the distribution of the load must be found according to the principles laid down in § 47.

Although it is not recommended to leave out the diagonals (except when the load is uniformly distributed, as in the trussed purlins of a roof), yet these theoretically imperfect structures occur so frequently that the influence of the omission will be illustrated by a few examples.

A theoretical parabolic girder of three bays should have one diagonal at least in the central bay (Fig. 410). Without this diagonal the deformation shown in Fig. 411 would occur

FIG. 410.

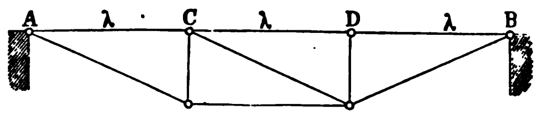
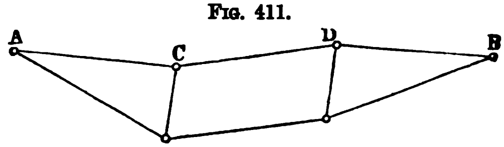


FIG. 411.



with a partial load. This can, however, be avoided by replacing the three bars A C, C D, and D B by a continuous stiff beam (Fig. 412). The structure is then no longer a simple truss, for even when fully loaded a part of the load will be carried directly by the stiff beam, and the remainder will be communicated by the two verticals to the rods A E, E F, and F B, and this part will be smaller the stiffer the beam is.

The structure consists of the two simple systems shown in Figs. 413 and 414; and to find the stresses produced by the two loads Q (Fig. 412), the ratio

$$\frac{Q_1}{Q_2} = \frac{n\,Q}{(1-n)\,Q}$$

must be determined by following the method indicated in § 47.

As soon as n is found, the distribution of a partial load can also be ascertained. Thus, for instance, if the point C alone is

FIG. 412.

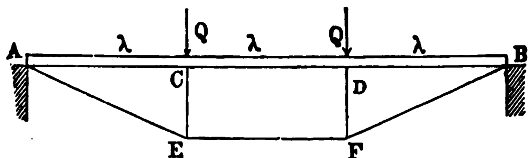


FIG. 413.

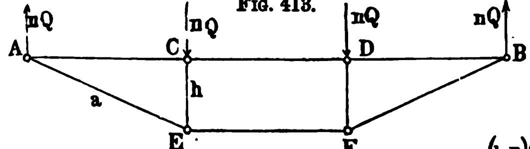
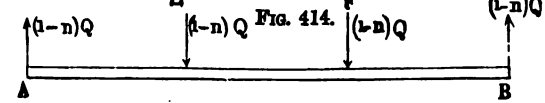


FIG. 414.



loaded with a weight Q (Fig. 415), the following distribution takes place. One part,

$$\frac{n}{2} Q \quad \text{and} \quad \frac{n}{2} Q,$$

forms the load on Fig. 416, and the remaining part,

$$\left(1 - \frac{n}{2}\right) Q \quad \text{and} \quad -\frac{n}{2} Q,$$

is the load on Fig. 417.

That this really is the case is easy to see, in the following way:—Evidently a load at D produces the same effect at C as a load at C produces at D (by symmetry). Now, since the horizontal stress in the bars A E and B F is always equal, the stress in both verticals must also always be equal. A single load at C therefore produces half the stress in the verticals that the two loads at C and D together do; or, in other words, the loads on C and D (Fig. 416) due to Q at C or at D are $\frac{n}{2}$ Q and $\frac{n}{2}$ Q. The loads on Fig. 417 are evidently found by subtraction.

The exterior forces on the structure being known, the stresses can be found by the method of moments; the stresses in the bars A E, E F, F B, due to the load at C, are given in

FIG. 415.

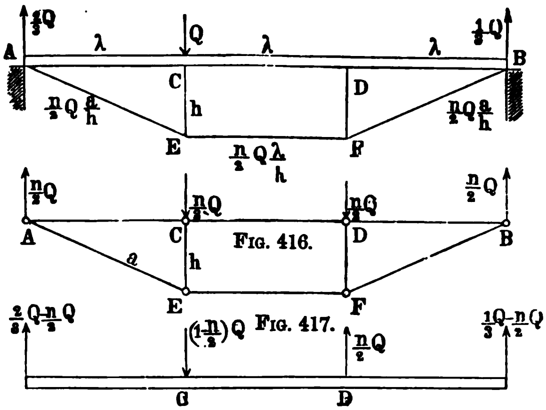


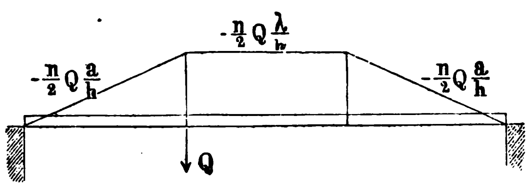
FIG. 416.

FIG. 417.

Fig. 415. These stresses will remain the same when the point D is loaded instead of the point C, and they will be doubled when both C and D are loaded.

The stresses in these bars, when the truss is fully loaded, are therefore to those that the same load would produce in a parabolic truss of the same form in the ratio of $n : 1$. Now the number n depends on the relative stiffness of each simple system, and can only be 1 when there are free joints at C and D. Thus it is evident that the error committed by treating

FIG. 418.

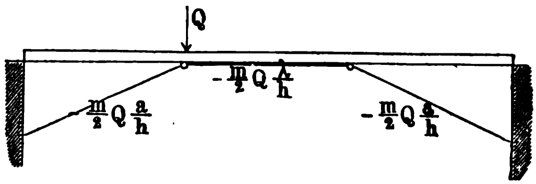


the structure of Fig. 412 as a simple parabolic truss is greater the smaller n is, or, in other words, the stiffer the beam is.

Obviously, if the truss be reversed (Fig. 418) the stresses

can be found in exactly the same manner. The stresses in the structure of Fig. 419 can be calculated in a similar manner, obtaining the number m from § 47, in the same way as n.

FIG. 419.



§ 49.—INFLUENCE OF CHANGES OF TEMPERATURE.

The results just obtained for composite structures are not practically useful, because the influence of the changes of temperature was not taken into account. As will be seen, the distribution of the load between the two simple systems, and also the economical ratio of their depths, depend on these variations of temperature.

It is found by experiment that all bodies, with one or two exceptions, expand as the temperature increases and contract when the temperature diminishes; and it is also found that the expansion or contraction is proportional to the increase or decrease of temperature. The ratio of the alteration of length due to one degree of temperature to the original length is called the coefficient of linear expansion. Thus, for wrought iron, if the temperature is reckoned in degrees Centigrade, this coefficient is

$$\alpha = 0\cdot0000122; \qquad [32]$$

that is to say, a wrought-iron bar increases or decreases its length by $\frac{122}{10,000,000}$ths for an increase or decrease of 1° C.

Since the alteration of length is proportional to the change of temperature, the ratio of the alteration of length due to t° of temperature to the original length will be

$$\Delta = \alpha t. \qquad [33]$$

(In other words, Δ is the amount which a unit of length is expanded or contracted by a change of temperature of t°.)

X

For example, if $t = + 41°$ C., the elongation of a wrought-iron bar is $= 41 \times \frac{122}{10.000.000} = \frac{1}{2000}$th of its original length.

If the increase of temperature were only 20·5° C., the elongation would only be $\frac{1}{4000}$th; and if the temperature were to decrease by the same amount, the consequent shortening would be the same as the previous elongation.

The alterations of form produced by variations of temperature are quite distinct from those due to elasticity. If, therefore, an increase of temperature coincides with tension in any bar, the increments of length due to both causes are to be added together to obtain the total elongation; and if a decrease of temperature occurs with compression, the separate decrements of length are to be added together. But the total elongation or shortening will be equal to the difference between the increment and the decrement, if compression occurs with increase of temperature, or tension with decrease of temperature.

In the combination of bars shown in Fig. 389 an increase of temperature of $t°$ would, according to equations 1 and 33, raise the point C by an amount

$$\sigma = \Delta \frac{a^2}{h}; \qquad [34]$$

and this point would sink by the same amount with a decrease of $t°$.

The braced arch of § 22 can, as far as the influence of temperature on the height of the hinge S is concerned, be regarded as a combination of two rods similar to the above. From the dimensions of the arch $h = 5000$ millimetres, $a^2 = 20000^2 + 5000^2$,* and assuming that $\Delta = \frac{1}{4000}$ (for an increase of 20·5° C.)

$$\sigma = \frac{1}{4000}\left(\frac{20000^2 + 5000^2}{5000}\right) = 21 \cdot 25 \text{ millimetres.}$$

Consequently the hinge S is raised 21·25 millimetres when an increase of temperature of 20·5° C. takes place, and therefore also sinks 21·25 millimetres with a decrease of 20·5° C. If the variations of temperature were doubled, the rising and the sinking of the hinge would be 42·5 millimetres. Supposing that the decrease of temperature of 41° C. occurred at the same time that the total load was on the bridge, which latter, according to § 45, equation 9, produces a deflection $s_1 = 18 \cdot 75$ millimetres, the total deflection would be

$$18 \cdot 75 + 42 \cdot 5 = 61 \cdot 25 \text{ millimetres.}$$

---

* See Fig. 175, page 127.

If δ_1 denotes the decrease in a unit of length due to compression in the two rods A E and B E (Fig. 401), produced by a load at C, and Δ the shortening per unit of length in the same bars owing to a coincident decrease of temperature, the total deflection of the point E is, according to equation 1:

$$s = (\delta_1 + \Delta)\frac{a^2}{h}, \qquad [35]$$

and this deflection is to be equated to $\frac{\delta_2 l^2}{f}$, the deflection in the centre of the horizontal beam (equation 18), thus:

$$\frac{(\delta_1 + \Delta)\,a^2}{h} = \frac{\delta_2\, l^2}{f}. \qquad [36]$$

Solving this equation for $\frac{f}{h}$, and putting $a^2 = l^2 + h^2$:

$$\frac{f}{h} = \left(\frac{\delta_2}{\delta_1 + \Delta}\right)\left(\frac{1}{1 + \frac{h^2}{l^2}}\right). \qquad [37]$$

The influence of the variations of temperature on the distribution of the load can be ascertained by finding the effect produced by the change of temperature on the structure, supposed weightless, and upon which no exterior loads are acting.

If Δ is the shortening per unit of length due to a decrease of temperature, the point E (Fig. 401), if free, would be lowered by an amount $\frac{\Delta\, a^2}{h}$, but the actual amount is less, owing to the horizontal beam C D; or, in other words, the stiffness of this beam produces a vertical force P acting upwards at E. Both struts are elongated by this force, and this elongation must be deducted from the shortening produced by the decrease of temperature. Therefore, if δ_1 is the elongation per unit of length produced by the force P alone, the actual depression of the point E is

$$s = (\Delta - \delta_1)\frac{a^2}{h}. \qquad [38]$$

But, according to equation 19, $\delta_1 = \frac{Pa}{2E_1 F_1 h}$. Hence:

$$s = \left(\Delta - \frac{Pa}{2E_1F_1h}\right)\frac{a^2}{h}. \qquad [39]$$

The deflection at the centre of the beam due to the force P is, from equation 14, $s = \frac{\delta_2 l^2}{f}$, and substituting $\delta_2 = \frac{Pl}{2E_2F_2f}$ from equation 20,

$$s = \frac{Pl^3}{2E_2F_2f^2}. \qquad [40]$$

Hence by equating the values of s in equations 39 and 40:

$$\left(\Delta - \frac{Pa}{2E_1F_1h}\right)\frac{a^2}{h} = \frac{Pl^3}{2E_2F_2f^2}. \qquad [41]$$

And solving for P:

$$P = \frac{2\Delta E_1 F_1 \frac{h}{a}}{1 + \frac{l^3}{a^3}\cdot\frac{E_1}{E_2}\cdot\frac{F_1}{F_2}\cdot\frac{h^2}{f^2}}. \qquad [42]$$

Evidently P is the correction to be applied to the distribution of the load between the two systems found from equation 21: in fact, a decrease of temperature diminishes the part of the load carried by the struts by the amount P, and consequently increases the load on the beam by the same amount. An increase of temperature has the reverse effect.

By substituting in the above equation $\frac{h}{a} = 0\cdot6$, $\frac{l}{a} = 0\cdot8$, $\frac{h}{f} = 3$, also $E_1 = E_2 = 20000$ (supposing that both systems are made of wrought iron), $F_1 = F_2 = 10000$ square millimetres, and $\Delta = \pm\frac{1}{4000}$ (corresponding to a change of temperature of 20·5° C.), the value of P is found to be $P = \pm 10700$ kilos.

By introducing the same data into equation 21, it is found that

$$\frac{Q_1}{Q_2} = 4\cdot608;$$

or if the total load is 80000 kilos.

$$Q_1 = \frac{\frac{Q_1}{Q_2}}{1 + \frac{Q_1}{Q_2}} Q = \frac{4 \cdot 608}{5 \cdot 608} \times 80000 = 65740 \text{ kilos.}$$

$$Q_2 = \frac{1}{1 + \frac{Q_1}{Q_2}} Q = \frac{1}{5 \cdot 608} \times 80000 = 14260 \text{ kilos.}$$

Thus, when the temperature decreases $20 \cdot 5^\circ$ C., the load carried by the struts becomes

$$65740 - 10700 = 55040 \text{ kilos.}$$

and by the beam

$$14260 + 10700 = 24960 \text{ kilos.}$$

And when the temperature increases $20^\circ \cdot 5$ C. the struts carry

$$65740 + 10700 = 76440 \text{ kilos.}$$

and the beam

$$14260 - 10700 = 3560 \text{ kilos.}$$

(This subject will be further discussed in the sixteenth chapter.)

FIFTEENTH CHAPTER.

§ 50.—Resistance of Beams to Flexure.

If one end of a horizontal beam be fixed, and a weight hung to the free end, the beam, originally straight, will be bent into the form of a curve, whose convexity is upwards (Fig. 420). If the beam be regarded as a bundle of fibres, whose direction is parallel to the length of the beam, and which are cemented together, so that they cannot slip over one another, it is easy to see that as soon as the beam is bent the upper fibres will be

Fig. 420.

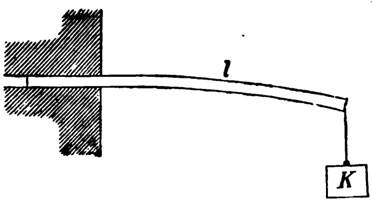
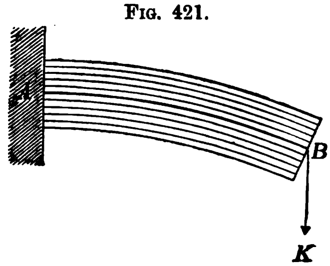


Fig. 421.



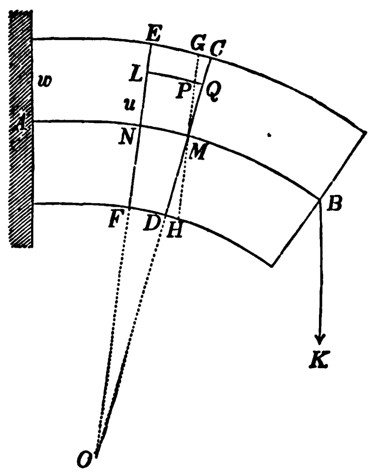
lengthened, and the lower ones shortened. Between the upper and the lower layer of fibres there must therefore be a layer in which the fibres are neither lengthened nor shortened; this layer A B (Fig. 421) can be called the *neutral surface.*

The greater the distance of a fibre from this neutral surface the greater will be its elongation if it is above the neutral surface, and the greater its shortening if below. It can be assumed that the sections made by planes at right angles to the neutral surface before bending remain plane after bending,* and are still at right angles to the neutral surface, which is now curved. In Fig. 422, M and N are two of these planes very near to each other; originally they were parallel, but when bending took

* This has been ascertained by experiment. See 'Civil Engineering,' by *Prof. Rankine.*—Trans.

place they converged in the directions C D and E F. Now, since the portions of the fibres lying between the two planes were originally all equal, the alteration in length of each fibre can be found by drawing a plane G H parallel to E F, and at a distance N M from it equal to the original length of the fibres, the distances between the planes C D and G H are evidently the alterations in length of the fibres. Hence from Fig. 422 for any fibre L Q:

FIG. 422.

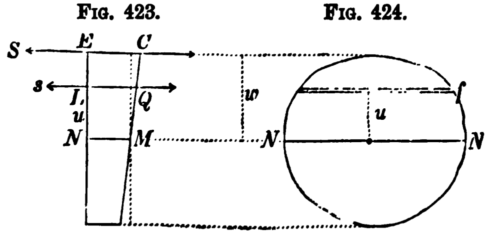


$$\frac{PQ}{GC} = \frac{u}{w};$$

or in words: *The alteration in length of any fibre is proportional to its distance from the neutral surface.* But, according to the laws of elasticity, the intensity of stress is proportional to the alteration of length, so long as the stress is below the limit of elasticity, hence the *stress in any fibre is proportional to its distance from the neutral surface so long as the stress in it is below the limit of elasticity.*

Therefore, if s is the stress (per unit of area) in a fibre L Q

FIG. 423. FIG. 424.



at a distance u from the neutral surface (Figs. 423 and 424), and if S is the stress (per unit of area) in the fibre E C, at a distance w:

$$\frac{s}{S} = \frac{u}{w}, \quad \text{or} \quad s = S\frac{u}{w}. \qquad [43]$$

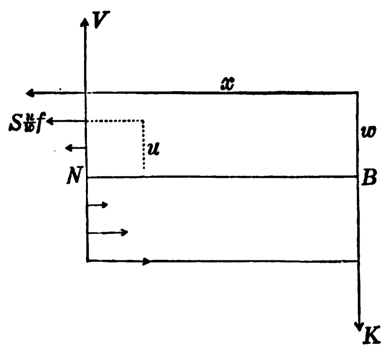
The total stress in any fibre can be found by multiplying its area by s, the stress per unit of area. If the beam be imagined divided into an infinite number of layers of fibres parallel to the neutral surface N N, and if f be the cross-sectional area of such a layer whose distance from N N is u, the total stress in this layer is evidently:

$$s f = S \frac{u}{w} f. \qquad [44]$$

This stress will be tension when the layer is above the neutral surface, that is, when u is positive; and will be compression when the layer is below the neutral surface, that is, when u is negative.

Let a section of the beam be made at N by a plane perpendicular to the neutral surface. For equilibrium a force will have to be applied to each fibre acting in the direction of its length, and equal to the stress in it (Fig. 425). These forces can be considered horizontal, for it must be assumed that the amount of bending is very small. They alone, however, could not maintain equilibrium; for, to resist the vertical force K acting downwards, a vertical force V, acting upwards, must be applied; and since the section plane N is very nearly vertical, V can be considered as acting along it. It is this force which prevents the part B N of the beam from slipping down along the section plane, and it is called the *resistance to shearing*. Resolving the forces acting on B N vertically:

FIG. 425.



$$V = K. \qquad [45]$$

The only horizontal forces acting on B N are the direct stresses in the fibres. Those above the neutral surface act from

right to left, and those below from left to right. The algebraical sum of these forces must be zero, therefore from equation 44:

$$\Sigma\left(\frac{S}{w} u f\right) = 0; \qquad [46]$$

and since $\frac{S}{w}$ is a common factor it can be omitted; hence:

$$\Sigma(f u) = 0. \qquad [47]$$

This equation shows that the sum of the products of the area of each elemental layer into its distance from the neutral surface is zero.

But, as is well known, if $\bar{x}$ be the distance of the centre of gravity from the neutral surface,

$$\bar{x}\Sigma(f) = \Sigma(f u) = 0, \quad \text{or} \quad \bar{x} = 0;$$

that is, *the neutral surface passes through the centre of gravity of the section.*

The third condition of equilibrium is that the sum of the moments about any axis should vanish. Let the intersection of the neutral surface with the section plane be this axis, which will be perpendicular to the paper at N (Fig. 425), and is called the *neutral axis.* The moment of K about this axis is K x (called the *moment of flexure* or else *bending moment*), and the moment of the stress in the layer of fibres, whose distance is u from the neutral surface, is $\frac{S u f}{w} u$. Hence:

$$\Sigma\left(\frac{S u f}{w} u\right) = K x. \qquad [48]$$

or in words: *The moment of resistance of the fibres is equal to the moment of flexure of the bending force.*

The common factor $\frac{S}{w}$ can be placed outside the sign of summation, thus

$$\frac{S}{w}\Sigma(f u^2) = K x. \qquad [49]$$

But the expression $\Sigma(f u^2)$ is the moment of inertia * of

* See footnote, page 301.

the cross-section about the neutral axis, and it is generally denoted by I. Therefore,

$$\frac{S}{w} I = K x. \qquad [50]$$

Or, writing M, instead of $K x$, to represent more generally the moment of flexure,

$$\frac{S}{w} I = M.^{*} \qquad [51]$$

S is the stress per unit of area in the fibre whose distance is w from the neutral surface. If, therefore, w is the greatest distance that occurs, S will be the greatest stress, and can be found from

$$S = \frac{w M}{I}; \qquad [52]$$

when

$$I = \Sigma (f u^2) \qquad [53]$$

is known.

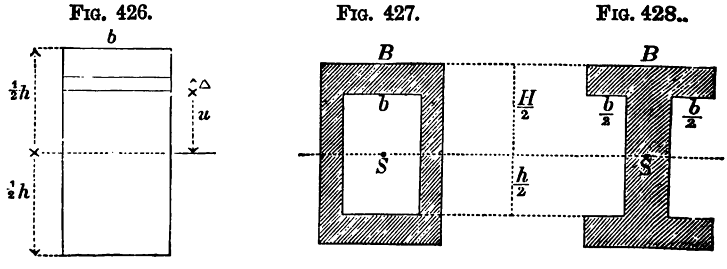
To find the value of I when the section is rectangular, consider a layer of fibres at a distance u from the neutral axis (Fig. 426), the area of this layer is $f = b \Delta$, and consequently,

$$I = b \Sigma (\Delta u^2);$$

or, according to the notation of the Integral Calculus:

$$I = b \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} u^2 \, d u = \frac{b h^3}{12}. \qquad [54]$$

FIG. 426. FIG. 427. FIG. 428.



The section shown in Fig. 427 can be considered as the dif-

* This equation has been obtained on the supposition that the stress nowhere exceeds the limit of elasticity. This formula is therefore not applicable at the point of rupture, and this is borne out by experiment.—TRANS.

ference of the two rectangular sections B H and $b\,h$, and hence its moment of inertia about the neutral axis is

$$I = \frac{B\,H^3}{12} - \frac{b\,h^3}{12}. \qquad [55]$$

The same value of I obtains for the section shown in Fig. 428. The moment of inertia of all sections that can be resolved into rectangles may be obtained in a similar manner by means of equation 54 so long as they are symmetrical with respect to the neutral axis. For instance, for the section shown in Fig. 429, which is the sum of two rectangles:

$$I = \frac{b\,h^3}{12} + \frac{b_1\,h_1^{\,3}}{12}; \qquad [56]$$

and for the section of Fig. 430:

$$I = \frac{B\,H^3}{12} - \frac{b\,h^3}{12} - \frac{b_1\,h_1^{\,3}}{12}. \qquad [57]$$

FIG. 429.

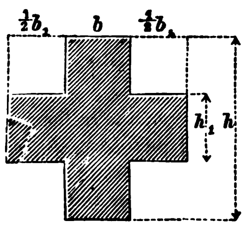
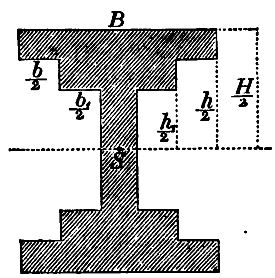


FIG. 430.



[NOTE.—It is sometimes necessary to find the moment of inertia of irregular figures whose contour is not determined by any simple curve, the section of a rail for instance.

In such cases the moment of inertia can only be obtained approximately by finding that of a figure composed of rectangles, as shown in Fig. 430A, and the greater the number of rectangles the nearer will be the approximation.

It becomes therefore necessary to know the moment of inertia of a rectangle about any axis parallel to one of its sides. Let the moment of inertia of the rectangle A B (Fig. 430B) be required about the axis Y O. Take an elemental strip P P′ of the rectangle, of width δx. The moment of inertia of

this element about Y O is $x^2 \cdot b\,\delta x$. The moment of inertia of the rectangle A B will therefore evidently be

$$= \int_a^c x^2 b\,d x$$

$$= \frac{b}{3}(c^3 - a^3).$$

FIG. 430A.

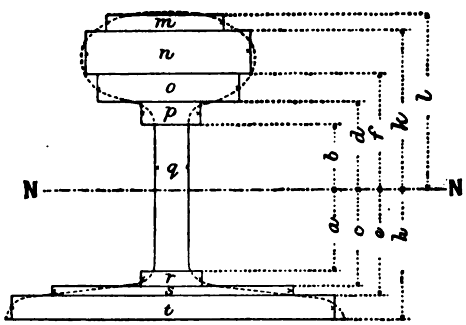
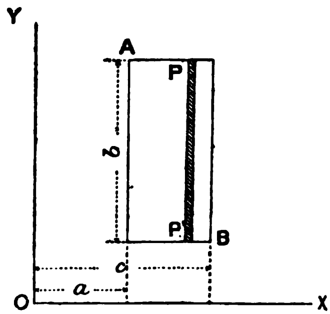


FIG. 430B.



Applying this to the case of Fig. 430A, the amount of inertia of that section about the axis N N is found to be

$$I = \tfrac{1}{3}\,\{\, m(l^3 - k^3) + n(k^3 - f^3) + o(f^3 - d^3) + p(d^3 - b^3) + q(b^3 + a^3) + v(c^3 - a^3) + s(e^3 - c^3) + t(h^3 - e^3)\,\}.$$

The axis N N passes through the centre of gravity of the section, and the position of this point must be found by the usual methods. A beautiful graphic method of finding the moment of inertia and the centre of gravity of *any* section is given in 'Graphic Statics,' translated from the German by Lieut. G. S. Clarke, R.E.]

If the section is circular (Fig. 431):

$$\Sigma(f u^2) = \Sigma(f v^2);$$

for evidently the moment of inertia of a circle is the same about any diameter; and since $u^2 + v^2 = x^2$,

$$\Sigma(f u^2) + \Sigma(f v^2) = \Sigma(f x^2);$$

or,

$$I = \tfrac{1}{2}\Sigma(f x^2).$$

$\Sigma(fx^2)$ can evidently be found by replacing f by the area of the elemental annulus (Fig. 432) whose radius is x, and breadth Δ. The area of this annulus is $2x\pi\Delta$; therefore,

$$\Sigma(fx^2) = 2\pi\Sigma(x^3\Delta);$$

Or in the notation of the Integral Calculus:

$$\Sigma(fx^2) = 2\pi\int_0^R x^3\,dx = \frac{\pi}{2}R^4;$$

FIG. 431.

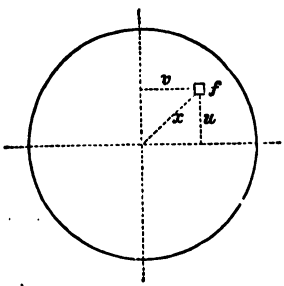
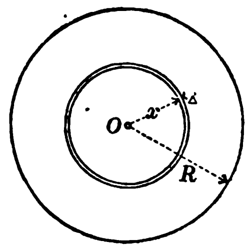


FIG. 432.



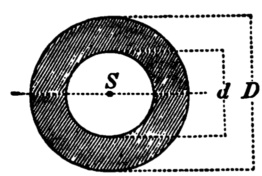
and if the diameter of the circle is D,

$$I = \frac{\pi}{4}R^4 = \frac{\pi}{64}D^4. \qquad [58]$$

The annular section shown in Fig. 433 can be regarded as the difference between two circles whose diameters are D and d respectively. Hence, in this case:

$$I = \frac{\pi}{64}(D^4 - d^4). \qquad [59]$$

FIG. 433.



If the section remains constant throughout the length of the beam, $\frac{I}{w}$ will also be constant, and it appears (from equation 50) that in this case the greatest intensity of stress will reach its greatest value in the beam at the point where the bending

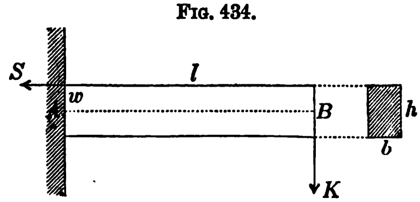
moment is amaximum. This is evidently the case when $x = l$; and if S_1 is the greatest stress in the beam:

$$\frac{S_1}{w} I = K l. \qquad [60]$$

If the beam has a rectangular section (Fig. 434), $w = \frac{h}{2}$, and from equation 54, $I = \frac{b h^3}{12}$; equation 60 therefore takes the form

$$\frac{S_1 b h^2}{6} = K l. \qquad [61]$$

FIG. 434.

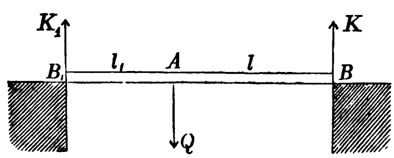


As an example, let $K = 125$ kilos., $l = 800$ millimetres, $b = 20$ millimetres, $h = 100$ millimetres, then the greatest stress in the beam is,

$$S_1 = \frac{6 K l}{b h^2} = \frac{6 \times 125 \times 800}{20 \times 100^2} = 3 \text{ kilos. per square millimetre.}$$

The intensity of the stress is independent of the nature of the material. By comparing it however with the stress per square millimetre considered safe for the material in question, it can be ascertained whether the resistance of the beam is sufficient or not, or to what extent the load might be increased so that the safe stress should just be arrived at. Thus if the above beam were made of wrought iron, the load K could be doubled (250 kilos.), in which case the greatest stress per square millimetre would be 6 kilos. (see page 264).

FIG. 435.



The greatest stress in a beam supported at both ends (Fig. 435), and loaded at any point A, can be found from equa-

tion 60 by writing for K the reaction at one of the abutments, and for l the distance of the load from that abutment. For if the part A B_1 be considered fixed (encased in a wall, for instance), it is evident that A B is in the same condition as the beam of Fig. 434, with this difference, however, that the bending force in this case acts upwards instead of downwards, and consequently the greatest tension occurs in the lowest fibre. The reaction K at the abutment A is found by taking moments about the other abutment, thus:

$$K(l + l_1) = Q\,l_1, \quad \text{or } K = \frac{Q\,l_1}{(l + l_1)};$$

and substituting in equation 60

$$\frac{S_1}{w} I = \frac{Q\,l\,l_1}{l + l_1} \qquad [62]$$

(Since $K_1\,l_1 = K\,l$ the same equation would be obtained if the greatest stress in the part A B_1 were found).

[NOTE.—If M is the bending moment at any section of the beam distant x from B_1, it is easily seen that

$$M = K\,x = \frac{Q\,l_1}{(l + l_1)}\,x,$$

so long as the section is situated between B_1 and A, and

$$M = \frac{Q\,l_1}{(l + l_1)}\,x - Q\,(x - l_1),$$

when the section lies between A and B.]

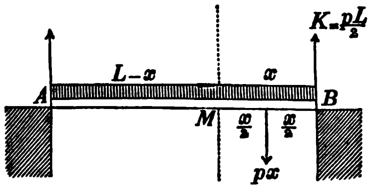
When a beam, supported at both ends, is loaded uniformly and p is the load per unit of length, the reaction at either abutment is $K = \frac{p\,L}{2}$, and $p\,x$ is the load on the part B M $= x$ (Fig. 436). If the part A M of the beam be imagined fixed the part B M becomes a beam fixed at one end and acted upon by two loads, namely, the reaction K acting upwards at B, and the load $p\,x$ acting downwards at the centre of B M. Taking moments about M:

$$M = K\,x - p\,x\frac{x}{2} = \frac{p\,L}{2}\,x - \frac{p\,x^2}{2}. \qquad [63]$$

Substituting in equation 51, it appears that the greatest stress in a section whose distance from one of the abutments is x can be found from the equation

$$M = \frac{S}{w} I = \frac{px}{2} (L - x). \qquad [63A]$$

FIG. 436.



The product x $(L - x)$ is greatest when $x = \frac{L}{2}$. Hence the greatest stress in the beam is given by the equation:

$$\frac{S_1}{w} I = \frac{p L^2}{8}. \qquad [64]$$

By putting $L = 2l$ and $x = l - z$ in equation 63, the general equation for the moment of flexure at any section is obtained, distances being measured from the centre of the beam, thus:

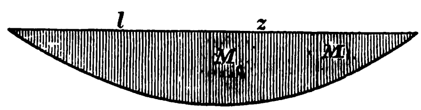
$$M = p \left(\frac{l^2 - z^2}{2}\right); \qquad [65]$$

and when $z = 0$:

$$M_0 = \frac{p l^2}{2}. \qquad [66]$$

Equation 65 can be exhibited graphically by taking the different values of z as abscissa, and the corresponding values

FIG. 437.



of M as ordinates. The curve shown in Fig. 437 is thus obtained.

The shearing force (Fig. 436) at the section through M is:

$$\frac{p\,L}{2} - p\,x = V;$$

for this is the vertical force that would have to be applied at this section to maintain equilibrium.

Putting as before $L = 2\,l$ and $x = l - z$:

$$V = p\,z. \qquad [66A]$$

§ 51.—Deflection of Beams.

The curved line A B (Fig. 422), into the form of which the neutral surface is bent, is called the curve of deflection.

The element of arc M N of this curve can be regarded as the arc of a circle whose centre is at O, the intersection of C D and E F. This circle is, in fact, the circle of curvature of the curve at M N, and ρ its radius is the radius of curvature. Since the two triangles C G M and M N O are similar:

$$\frac{C\,G}{M\,N} = \frac{M\,G}{O\,N}.$$

Now $\frac{C\,G}{M\,N}$ is evidently the elongation per unit of length of the fibre E C, the stress in which is S per unit of area, therefore, according to equation I:

$$\frac{C\,G}{M\,N} = \frac{S}{E}.$$

Further $M\,G = w$ and $O\,N = \rho$. Hence:

$$\frac{S}{E} = \frac{w}{\rho}, \quad \text{or} \quad \frac{S}{w} = \frac{E}{\rho}. \qquad [67]$$

Substituting this value of $\frac{S}{w}$ in equation 51:

$$\frac{E\,I}{\rho} = M. \qquad [68]$$

The radius of curvature of the element of arc M N of the deflection curve A B (Fig. 438) is therefore:

$$\rho = \frac{E\,I}{M} = \frac{E\,I}{K\,(l - x)}. \qquad [69]$$

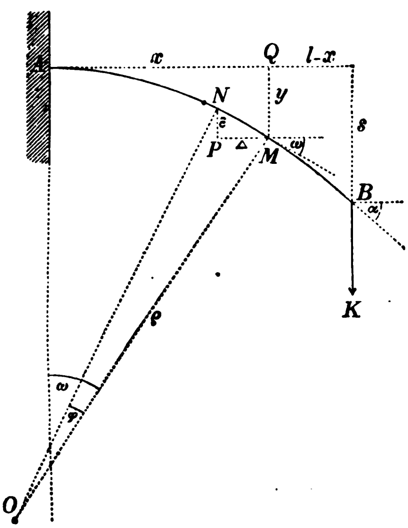
As already pointed out (p. 312) the curvature is assumed to be very small, and consequently instead of the arc $\widehat{MN}$ in the equation

$$\widehat{MN} = \rho\phi, \quad \text{or} \quad \phi = \frac{\widehat{MN}}{\rho},$$

its horizontal projection $MP = \Delta$ can be substituted. Therefore:

$$\phi = \frac{K(l-x)\Delta}{EI}. \qquad [70]$$

Fig. 438.



Now, consider the horizontal projection $AQ = x$ of the arc AM resolved into its elements Δ, and let x be replaced in the above equation by its successive values differing each by Δ; by adding together all the equations thus obtained the sum of all the small angles ϕ will be found, thus:

$$\Sigma(\phi) = \Sigma\left(\frac{K(l-x)\Delta}{EI}\right). \qquad [71]$$

$\Sigma(\phi)$ is evidently ω, the angle which the radius of curvature MO makes with the vertical (or the angle the tangent to the

curve at M makes with the horizontal). Let the beam be prism, that is, let its cross-section be the same throughout; then I is constant and $\frac{K}{EI}$ is a common factor of all the terms contained under the sign of summation in the right-hand expression. Hence:

$$\omega = \frac{K}{EI} \Sigma [(l - x\,\Delta)] = \frac{K}{EI} [l\,\Sigma(\Delta) - \Sigma(x\,\Delta)]. \qquad [72]$$

but

$$\Sigma(\Delta) = x \quad \text{and} \quad \Sigma(x\,\Delta) = \frac{x^2}{2}.$$

Therefore:

$$\omega = \frac{K}{EI}\left(l x - \frac{x^2}{2}\right). \qquad [73]$$

If $x = l$, ω becomes a, the angle the tangent to the curve at the end B of the beam makes with the horizontal, and

$$a = \frac{K\,l^2}{2\,EI}. \qquad [74]$$

In the triangle M N P

$$\epsilon = \Delta \tan \omega;$$

but since the curvature is small, ω is a small angle, and consequently

$$\tan \omega = \omega,$$

or

$$\epsilon = \omega\,\Delta. \qquad [75]$$

Substituting in equation 73

$$\epsilon = \frac{K}{EI}\left(l x - \frac{x^2}{2}\right)\Delta; \qquad [76]$$

again replacing x by its successive values differing from each other by Δ, and summing up the values of ϵ thus found:

$$\Sigma(\epsilon) = \frac{K}{EI}[l\,\Sigma(x\,\Delta) - \tfrac{1}{2}\Sigma(x^2\,\Delta)]. \qquad [77]$$

Now $\Sigma(\epsilon)$ is evidently equal to y, the vertical projection of the arc A M, and further,

$$\Sigma(x\,\Delta) = \frac{x^2}{2}, \quad \Sigma(x^2\,\Delta) = \frac{x^3}{3};$$

therefore equation 77 becomes:

$$y = \frac{K}{EI}\left(\frac{l x^2}{2} - \frac{x^3}{6}\right). \qquad [78]$$

According to the notation of the Differential Calculus,

$$\Delta = d x, \quad \epsilon = d y, \quad \omega = \frac{d y}{d x}, \quad \text{and} \quad \phi = d \omega = d \left(\frac{d y}{d x}\right).$$

Equation 70 therefore takes the form

$$EI \frac{d^2 y}{d x^2} = K (l - x).$$

Assuming as before that I is constant, integrating twice and remembering that when $x = o$, $\frac{d y}{d x} = o$, and $y = o$, so that the constant occurring in each integration is zero.

$$EI \frac{d y}{d x} = K \left(l x - \frac{x^2}{2}\right)$$

$$EI y = K \left(\frac{l x^2}{2} - \frac{x^3}{6}\right).$$

From a reference to Fig. 438 it will be seen that when $x = l$, $y = s$, and therefore the deflection of the end B is (equation 78):

$$s = \frac{K l^3}{3 EI}. \qquad [79]$$

Dividing this last equation by equation 60,

$$\frac{s w}{S} = \frac{l^2}{3 E};$$

and substituting $\frac{S}{E} = \delta$, and $2 w = h$, (assuming that the centre of gravity of the cross-section is equidistant from the top and bottom fibres),

$$s = \tfrac{2}{3} \delta \frac{l^2}{h}.$$

It is evident that this equation is also true in the case of the beam represented by Fig. 435, when the weight Q is hanging at the centre. This equation can therefore be employed to find what alteration will be produced in the equation found for the compound system of Fig. 401, when the beam, instead of being of uniform strength, has an equal section throughout. Thus, putting $\frac{2}{3} \delta_2$, instead of δ_2,

$$\frac{\delta_1 a^2}{h} = \tfrac{2}{3} \delta_2 \frac{l^2}{f},$$

and equation 21 becomes

$$\frac{Q_1}{Q_2} = \tfrac{2}{3} \cdot \frac{l^3}{a^3} \cdot \frac{h^2}{f^2} \cdot \frac{E_1}{E_2} \cdot \frac{F_1}{F_2}.$$

[NOTE.—The equation

$$\frac{\mathrm{CG}}{\mathrm{MN}} = \frac{\mathrm{MG}}{\mathrm{ON}}$$

is quite independent of the manner in which the beam is loaded or supported, and is in fact a geometrical property depending only on the curvature. Equation 68, viz.

$$\frac{\mathrm{EI}}{\rho} = \mathrm{M}$$

is therefore perfectly general and is applicable to the case of any beam under bending stress.

Now, it is shown in works on geometry that for any curve referred to rectangular axes,

$$\rho = \pm \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2 y}{dx^2}}.$$

Let the straight line through the abutments be taken as the axis of x, then the tangent of the angle the tangent to the curve at the point (x, y) makes with the axis of x is $\frac{dy}{dx}$. But by a previous assumption the curvature is very small, and therefore this angle will also be very small for every point of the deflection curve; consequently $\left(\frac{dy}{dx}\right)^2$ may be neglected in comparison to 1. Hence in the present case,

$$\rho = \pm \frac{1}{\frac{d^2 y}{dx^2}}.$$

Therefore,

$$\frac{d^2 y}{dx^2} = \pm \frac{\mathrm{M}}{\mathrm{EI}}.$$

The positive or negative sign being taken according as $\frac{dy}{dx}$ increases or diminishes with x.

This is the differential equation to the deflection curve, and when M and I are known for any point (x, y), the equation to the curve can be found if the integrations can be effected.]

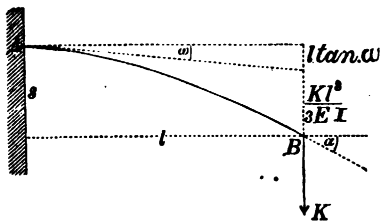
The beam instead of being horizontal before being bent may make a very small angle ω with the horizontal, Fig. 439. In this case the angle a the tangent at B makes with the horizontal, consists of two parts; one part is the angle ω and the other is the deflection angle that would obtain if the beam

had been originally horizontal, and which can be found from equation 74. Therefore:

$$\alpha = \omega + \frac{K l^2}{2 E I}. \qquad [80]$$

In the same manner the deflection s can be considered as

FIG. 439.

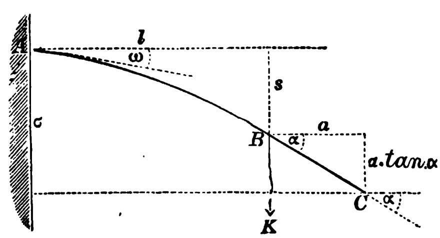


made up of two parts; one part is $l \tan \omega = l \omega$ (since ω is a small angle) and the other part can be obtained from equation 79. Hence:

$$s = l \omega + \frac{K l^3}{2 E I}. \qquad [81]$$

The above equations (80 and 81) can also be adapted to the case when the loaded point B is not at the end of the beam. If, in Fig. 440, C is the end of the beam, the part B C will remain straight if there be no load on it. The angle made with

FIG. 440.



the horizontal will be the same at C as at B, and is consequently equal to α (equation 80). The deflection at C is equal to the deflection at B added to $a \tan \alpha = a\,\alpha$ (since α is a small angle). Therefore:

$$\sigma = l \omega + \frac{K l^3}{3 E I} + a\,\alpha. \qquad [82]$$

And substituting for a its value from equation 80:

$$\sigma = (l + a)\,\omega + \frac{K(2l^3 + 3al^2)}{6EI}. \qquad [83]$$

Or adopting the notation of Fig. 441, that is, writing x instead of l, $l - x$ instead of a, and s instead of σ:

$$\alpha = \omega + \frac{Kx^2}{2EI}. \qquad [84]$$

$$s = l\omega + \frac{K(3lx^2 - x^3)}{6EI}. \qquad [85]$$

If there is also a load at C (Fig. 442), the deflection-angle and the deflection at C will be increased by an amount which can be found from equations 74 and 79, thus:

$$\alpha = \omega + \frac{Kx^2}{2EI} + \frac{Ql^2}{2EI}. \qquad [86]$$

$$s = l\omega + \frac{K(3lx^2 - x^3)}{6EI} + \frac{Ql^3}{3EI}. \qquad [87]$$

FIG. 441.

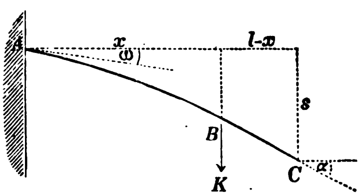
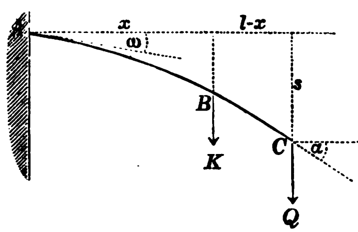


FIG. 442.



These equations are evidently also true if K and Q act upwards instead of downwards, when, however, their signs must be changed.

In the last equations put $\omega = 0$, $Q = 0$ and $p\,dx$ instead of K, further $d\alpha$ instead of α and ds instead of s, then:

$$d\alpha = \frac{px^2\,dx}{2EI}. \qquad [88]$$

$$ds = \frac{p(3lx^2 - x^3)\,dx}{6EI}. \qquad [89]$$

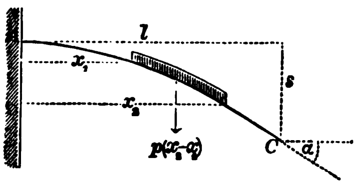
These equations give the effect, on a beam originally horizontal, of the element $p\,d\,x$ of a distributed load.

Integrating between the limits $x = x_1$ and $x = x_2$ for the case shown in Fig. 443 :*

$$a = \frac{p\,(x_2^3 - x_1^3)}{6\,\mathrm{E\,I}}. \qquad [90]$$

$$s = \frac{p\,[l\,(x_2^3 - x_1^3) - \frac{1}{4}\,(x_2^4 - x_1^4)]}{6\,\mathrm{E\,I}}. \qquad [91]$$

FIG. 443.



If the whole span is loaded with a load p per unit of length the limits of the integration are $x_1 = 0$, $x_2 = l$ and then :

$$a = \frac{p\,l^3}{6\,\mathrm{E\,I}}. \qquad [92]$$

$$s = \frac{p\,l^4}{8\,\mathrm{E\,I}}. \qquad [93]$$

Lastly, if the original inclination to the horizontal at the point of fixing A is ω and there is a load Q at C besides the uniformly distributed load:

$$a = \omega + \frac{p\,l^3}{6\,\mathrm{E\,I}} + \frac{\mathrm{Q}\,l^2}{2\,\mathrm{E\,I}}. \qquad [94]$$

$$s = l\,\omega + \frac{p\,l^4}{8\,\mathrm{E\,I}} + \frac{\mathrm{Q}\,l^3}{3\,\mathrm{E\,I}}. \qquad [95]$$

FIG. 444.

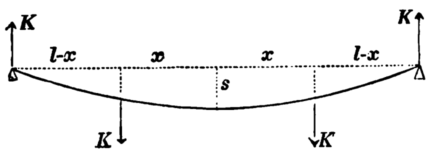


Fig. 444 represents a beam supported at both ends. If the left half be considered fixed in a wall, it is evident that the right

* The load is supposed to be uniformly distributed so that p is a constant.—TRANS.

half is in the same condition as the fixed beam of Fig. 442, when $\omega = 0$ and $Q = -K$, and the deflection can therefore be obtained by substituting these values of ω and Q in equation 87 and writing $-s$ instead of s, thus:

$$s = \frac{K(2l^3 - 3lx^2 + x^3)}{6EI}. \qquad [96]$$

[NOTE.—If $x = 0$ the beam is loaded with a central load 2 K, and

$$s = \frac{2Kl^3}{6EI};$$

or replacing 2 K by Q,

$$s = \frac{Ql^3}{6EI}.$$

The equation to the deflection curve of a beam supported at both ends and loaded in the centre, can be found from the equation given at p. 325, namely:

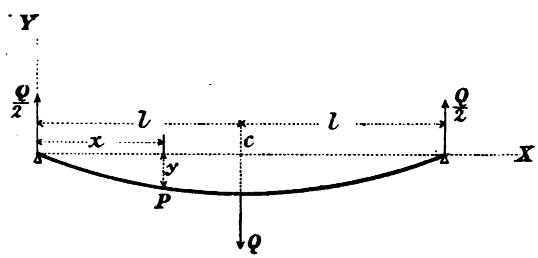
$$\frac{d^2y}{dx^2} = \pm\frac{M}{EI}.$$

If the beam is of the same section throughout, E I is constant, and (Fig. 444A) taking O X and O Y as axes of reference, the moment of flexure at any point P is

$$M = \frac{Qx}{2},$$

so long as x is less than l.

FIG. 444A.



Hence:

$$\frac{d^2y}{dx^2} = -\frac{Q}{2EI}x.$$

The negative sign being taken because $\frac{dy}{dx}$ diminishes as x increases.

Integrating

$$\frac{dy}{dx} = -\frac{Q}{4EI}x^2 + \text{constant}.$$

Now, when $x = l$, $\frac{dy}{dx} = 0$, because the tangent to the curve is horizontal at the centre of the beam. Therefore:

$$0 = -\frac{Q}{4EI} l^2 + \text{constant};$$

or

$$\frac{dy}{dx} = \frac{Q}{4EI}(l^2 - x^2).$$

Integrating again,

$$y = \frac{Q}{4EI}\left(l^2 x - \frac{x^3}{3}\right) + \text{constant}.$$

but when $y = 0$, $x = 0$, hence constant $= 0$ and

$$y = \frac{Q}{4EI}\left(l^2 x - \frac{x^3}{3}\right).$$

—the equation required. It will be observed that this equation is only true up to the centre of the beam, for at this point M changes from $\frac{Q\,x}{2}$ to $\frac{Q\,x}{2} - Q\,(x - l)$; but evidently the two halves of the deflection curve are exactly similar.

The greatest deflection occurs at the centre of the beam where $x = l$, and writing s for the greatest value of y,

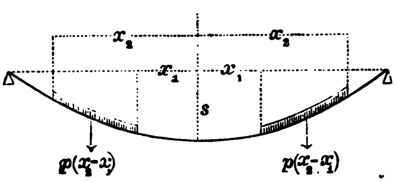
$$s = \frac{Q\,l^3}{6EI};$$

or the value found above.]

The deflection due to a uniformly distributed load can be found from equation 96 by first finding the deflection due to an element of load $p\,d\,x$; thus writing $d\,s$ instead of s, and $p\,d\,x$ instead of K:

$$d\,s = \frac{p\,(2\,l^3 - 3\,l\,x^2 + x^3)\,d\,x}{6EI}. \qquad [97]$$

FIG. 445.



Then the deflection at the centre in Fig. 445 is obtained by integrating this equation between the limits x_1 and x_2, thus:

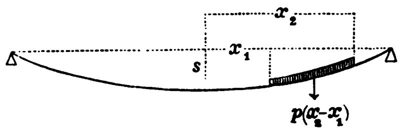
$$s = \frac{p\,[2\,l^3(x_2 - x_1) - l\,(x_2^3 - x_1^3) + \frac{1}{4}(x_2^4 - x_1^4)]}{6EI}. \qquad [98]$$

If the whole span be loaded, the limits of the integration are $x_1 = 0$ and $x_2 = l$, in this case, therefore:

$$s = \frac{5}{24}\frac{p\,l^4}{EI}. \qquad [99]$$

Since the load on the left half of the beam produces the same deflection as that on the right half, it is obvious that if the load on one side be removed the deflection in the centre will be

FIG. 446.

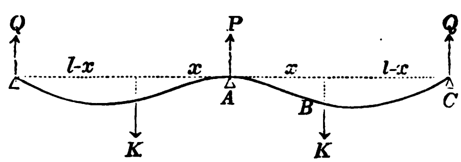


halved. The deflection at the centre of a beam loaded as indicated in Fig. 446 will therefore be:

$$s = \frac{p\left[2\,l^2(x_2 - x_1) - l(x_2^3 - x_1^3) + \frac{1}{4}(x_2^4 - x_1^4)\right]}{12\,EI}. \qquad [100]$$

If the left half of a beam supported at three equidistant points as shown in Fig. 447 be imagined fixed in a wall, the

FIG. 447.



right half is evidently in the same condition as the beam of Fig. 442 when $\omega = 0$, $s = 0$, and Q acts upwards or is negative.

By substituting these values in equation 87,

$$0 = \frac{K(3\,l\,x^2 - x^3)}{6\,EI} - \frac{Q\,l^3}{3\,EI};$$

and solving for Q,

$$Q = \frac{K}{2}\left(3\frac{x^2}{l^2} - \frac{x^3}{l^3}\right). \qquad [101]$$

Further the sum of the reactions at the three points of support must equal 2 K. Hence:

$$P = 2K - 2Q = 2K - K\left(3\frac{x^2}{l^2} - \frac{x^3}{l^3}\right). \qquad [102]$$

Thus the reactions at the three points of support are found.

[NOTE.—Obviously the three points of support are in a straight line, for it was assumed that $s = 0$. It is important to remember this, for the reactions depend on the relative positions of the three points of support.]

Evidently the load on the left span produces the same reaction at the central support as the load on the right span. Therefore in the case represented in Fig. 448, the reaction W at the central support is:

$$W = \tfrac{1}{2} P = K - Q; \qquad [103]$$

or

$$W = K\left\{1 - \tfrac{3}{2}\frac{x^2}{l^2} + \tfrac{1}{2}\frac{x^3}{l^3}\right\}. \qquad [104]$$

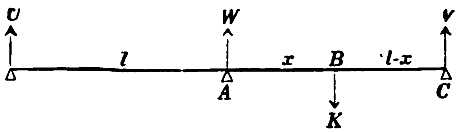
The reactions of the end supports can now be obtained by taking moments, thus:

$$V = K\left(\frac{l+x}{2l}\right) - \frac{W}{2},$$

and

$$U = K\left(\frac{l-x}{2l}\right) - \frac{W}{2};$$

FIG. 448.



or substituting for W its value:

$$V = \frac{K}{4}\left\{2\frac{x}{l} + 3\frac{x^2}{l^2} - \frac{x^3}{l^3}\right\} \qquad [105]$$

$$U = -\frac{K}{4}\left\{2\frac{x}{l} - 3\frac{x^2}{l^2} + \frac{x^3}{l^3}\right\}. \qquad [106]$$

The bending moment at a point N (Fig. 449), between A and B, and situated at a distance z from A, is:

$$M = K(x - z) - V(l - z). \qquad [107]$$

Putting $M = 0$, substituting for V, and solving for z:

$$\frac{z}{l} = \frac{2\frac{x}{l} - \frac{x^2}{l^2}}{4 + 2\frac{x}{l} - \frac{x^2}{l^2}}; \qquad [108]$$

which, when

$$\frac{l-x}{l} = u, \text{ and } \frac{l-z}{l} = v,$$

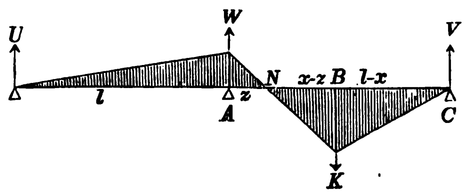
becomes

$$v = \frac{4}{5-u^2}. \qquad [109]$$

This equation determines the position of the point where the bending moment is zero.

The value of the bending moment for every point in the beam is represented graphically in Fig. 449, for the bending moment at every point from A to B is given by equation 107,

FIG. 449.



and this equation represents a straight line, M and z being the variables.

Let it be supposed that the beam is subject to a uniformly distributed moving load, then equation 109, when put in the form

$$u = \sqrt{5 - \frac{4}{v}}, \qquad [110]$$

FIG. 450.

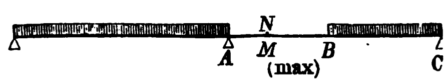
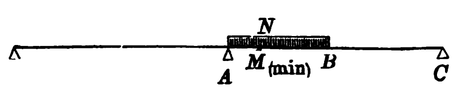


FIG. 451.



determines what parts of the beam should be loaded to produce a maximum or a minimum bending moment at N (see Figs. 450 and 451).

[NOTE.—The truth of this can be shown as follows:—It will be found that by substituting for V its value from equation 105, and putting as before $\frac{l-x}{l} = u$ and $\frac{l-z}{l} = v$, that the equation

$$M = K(x - z) - V(l - z)$$

becomes:

$$M = \frac{ulK}{4v}\left(5 - \frac{4}{v} - u^2\right).$$

Therefore M is positive so long as

$$u^2 < 5 - \frac{4}{v},$$

or

$$u < \sqrt{5 - \frac{4}{v}}.$$

Thus, evidently any load situated between C and B produces a positive bending moment at N. A little consideration will show that all loads on the other span also produce positive bending moments at N. Hence Fig. 450.

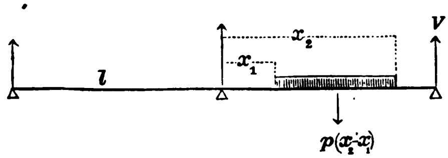
Again M is negative when

$$u > \sqrt{5 - \frac{4}{v}};$$

that is, when the load is placed between B and A. Therefore the greatest negative or minimum bending moment occurs at N when the part B A is loaded as in Fig. 451.]

When the reactions produced by a single load K have been found by means of equations 104, 105, and 106, the reactions due

FIG. 452.



to a uniformly distributed load can be obtained by writing $p\,dx$ instead of K and integrating between the limits x_1 and x_2 (Fig. 452). Thus:

$$V = \int_{x_1}^{x_2} \frac{p\,dx}{4}\left\{2\frac{x}{l} + 3\frac{x^2}{l^2} - \frac{x^3}{l^3}\right\};$$

or

$$V = \frac{p}{4\,l^3}\left\{l^2(x_2^2 - x_1^2) + l(x_2^3 - x_1^3) - \tfrac{1}{4}(x_2^4 - x_1^4)\right\}. \quad [111]$$

If several portions of the girder are thus loaded, the total reaction is found by adding together the reactions produced by each part separately.

Again, when the load is uniformly distributed over both spans, the reaction at either outer support can be found if $p\,dx$ be written instead of K in equation 101 and the integration performed between the limits 0 and l, thus:

$$Q_1 = \frac{p}{2}\int_0^l \left(\frac{3\,x^2}{l^2} - \frac{x^3}{l^3}\right) dx = \tfrac{3}{8}\,p\,l, \qquad [111\text{A}]$$

and

$$P_1 = 2\,p\,l - 2\,.\,\tfrac{3}{8}\,p\,l = \tfrac{5}{4}\,p\,l. \qquad [111\text{B}]$$

If, therefore, p is the uniformly distributed load per unit of length, and m is a uniformly distributed moving load on the part $x_2 - x_1$ of the beam represented in Fig. 452, the reaction V is,

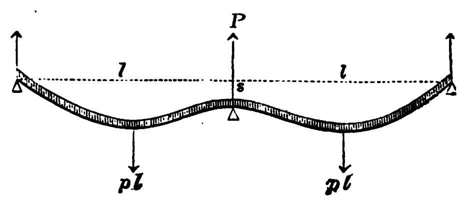
$$V = \tfrac{3}{8}\,p\,l + \frac{m}{4\,l^3}\left\{ l^2(x_2{}^2 - x_1{}^2) + l\,(x_2{}^3 - x_1{}^3) - \tfrac{1}{4}(x_2{}^4 - x_1{}^4) \right\}. \qquad [112]$$

Fig. 453 represents a beam uniformly loaded and supported at three equidistant points, the central support being at a distance s from the horizontal straight line joining the outer points of support.

Supposing the central support to be removed, the deflection s at the centre of the beam would be found from equation 99.

$$s_1 = \tfrac{5}{24}\,\frac{p\,l^4}{\mathrm{E\,I}}. \qquad [113]$$

FIG. 453.



Again, the upward deflection s_2 produced by an upward force P acting on the unloaded beam, can be found from equation 96 by putting $x = 0$ and $2\,K = P$, thus;

$$s_2 = \frac{P\,l^3}{6\,\mathrm{E\,I}}. \qquad [114]$$

Now s is evidently equal to the difference of these deflections, therefore

$$s = s_1 - s_2 = \tfrac{5}{24}\frac{p\,l^4}{EI} - \frac{P\,l^3}{6EI}. \qquad [115]$$

[NOTE.—This result can also be obtained as follows:
If, in equation, 95 $-$ Q be written for Q, $-s$ for s, and $\omega = 0$,

$$-s = \frac{p\,l^4}{8EI} - \frac{Q\,l^3}{3EI}.$$

This equation evidently applies to either half of the beam in Fig. 453, also

$$2Q + P = 2pl, \qquad \text{or} \qquad Q = pl - \frac{P}{2}.$$

Therefore,

$$-s = \frac{p\,l^4}{8EI} - \frac{p\,l^4}{3EI} + \frac{P\,l^3}{6EI}$$

$$s = \tfrac{5}{24}\frac{p\,l^4}{EI} - \frac{P\,l^3}{6EI}.$$

Equation 115 can be used to find the distribution of the load in Fig. 401, when the beam has the same cross-section throughout and is uniformly loaded. For the force P, taken in the opposite direction, deflects the apex of the struts by an amount

$$s = \frac{P\,a^3}{2E_1F_1h_1^2}; \qquad [116]$$

and by equating these two values of s

$$\frac{P\,a^3}{2E_1F_1h_1^2} = \tfrac{5}{24}\frac{p\,l^4}{E_2\,I_2} - \frac{P\,l^3}{6E_2I_2}. \qquad [117]$$

If the cross-section of the beam is rectangular, F_2 its area, and h_2 its height,

$$I_2 = \frac{F_2\,h_2^2}{12},$$

according to equation 54, and the above equation solved for P becomes

$$P = \frac{5pl}{4 + \frac{a^3}{l^3}\cdot\frac{h_2^2}{h_1^2}\cdot\frac{E_2}{E_1}\cdot\frac{F_2}{F_1}}. \qquad [118]$$

This equation gives the part of the load supported by the struts, supposing that the temperature does not alter and that

originally, before loading, the three points C, E, D were in a straight line.

If, for instance, $\frac{a}{l} = \frac{5}{4}$, $\frac{h_2}{h_1} = \frac{1}{3}$, $\frac{E_2}{E_1} = 1$, and $\frac{F_2}{F_1} = 2$, the equation gives

$$P = 1 \cdot 128\, p\, l.$$

If $\frac{F_2}{F_1} = 0$, $P = \frac{5}{4}\, p\, l$, or the same value that was obtained (equation 111 B) for the reaction at the central support of a uniformly loaded beam resting on three equidistant points placed in a horizontal straight line.*

When the temperature decreases the portion of the load carried by the beam is increased, the struts being relieved of the same amount, and the reverse occurs when the temperature increases. This "temperature load" P can be found by equating the deflection of the centre of the beam (equation 39) to the deflection of the apex of the struts (equation 114), thus:

$$\left(\Delta - \frac{P\, a}{2\, E_1\, F_1\, h_1}\right) \frac{a^2}{h_1} = \frac{P\, l^3}{6\, E_2\, I_2}. \qquad [119]$$

Substituting for $I = \frac{F_2\, h_2^{\,2}}{12}$, and solving for P,

$$P = \frac{2\, \Delta\, E_1\, F_1\, \frac{h_1}{a}}{1 + 4\, \frac{E_1}{E_2} \cdot \frac{F_1}{F_2} \cdot \frac{l^3}{a^3} \cdot \frac{h_1^{\,2}}{h_2^{\,2}}}. \qquad [120]$$

If $E_1 = E_2 = 20{,}000$ kilos., $F_1 = 10{,}000$ square millimetres, $F_2 = 20{,}000$ square millimetres, $\frac{l}{a} = 0 \cdot 8$, $\frac{h_1}{a} = 0 \cdot 6$, $\frac{h_1}{h_2} = 3$, and $\Delta = \frac{1}{4000}$ (corresponding to a decrease of temperature of $20^\circ \cdot 5$ C.), it will be found that $P = 5873$ kilos. And if at the same time the beam is subject to a uniformly distributed load $2\, p\, l$ (see equation 118),

$$P = 1 \cdot 128\, p\, l - 5873 \text{ kilos.}$$

for a decrease of $20^\circ \cdot 5$ C.; and similarly

$$P = 1 \cdot 128\, p\, l + 5873 \text{ kilos.}$$

for an increase of $20^\circ \cdot 5$ C.

It was shown, page 321, that the general equation for the radius of curvature of the elastic curve is

$$\rho = \frac{E\, I}{M}, \qquad [121]$$

---

* Since $\frac{F_2}{F_1} = 0$, the resistance of the struts is infinite, and they therefore act as a fixed point of support.—TRANS.

where I is the moment of inertia, and M the bending moment for the cross-section at which ρ is taken.

Thus when the ratio $\frac{I}{M}$ is constant for all sections ρ is constant, and the *deflection* curve is a circle.

Now in a beam of equal section throughout, I is constant. A prismatic beam can therefore only bend in the shape of a circle when M is constant. This would be the case, for instance, in Fig. 422, if instead of the single force K, a couple acted at the free end of the beam.

If, however, both M and I vary, and M_1, I_1 are the moments of bending and inertia respectively at any given section, for instance at one of the ends of the beam, then the equation

$$\frac{I}{M} = \frac{I_1}{M_1} \qquad [122]$$

or

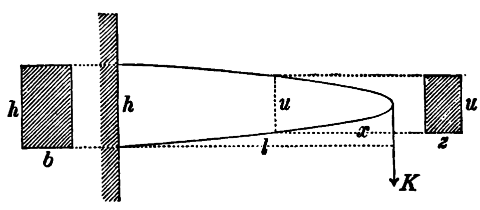
$$\frac{I}{I_1} = \frac{M}{M_1}$$

expresses the relation that must exist in order that the deflection curve may be a circle.

For example, in the case represented in Fig. 454, the general condition takes the form,

$$\frac{z u^3}{b h^3} = \frac{x}{l}. \qquad [123]$$

FIG. 454.



If $z = b$, as in Figs. 455 and 456, the equation to the curve to which the beam must be formed in elevation is

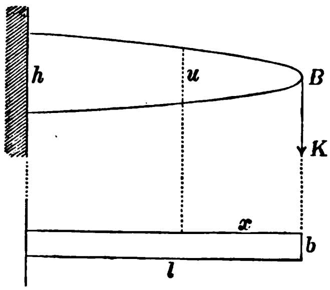
$$\frac{u^3}{h^3} = \frac{x}{l}. \qquad [124]$$

Again, if $u = h$,

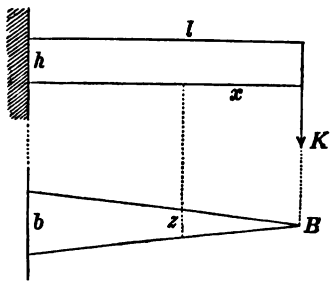
$$\frac{z}{b} = \frac{x}{l}, \qquad [125]$$

showing that the beam should be triangular on plan when of constant height.

FIGS. 455 AND 456.



FIGS. 457 AND 458.



The deflection of the point of loading can be found from the equation to the circle, viz. (Fig. 459),

$$l^2 = 2\,\rho\,s - s^2.$$

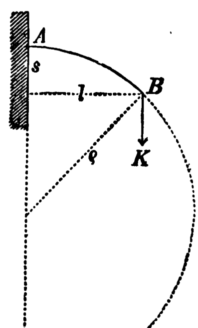
But since the amount of bending is very small, s^2 can be neglected and l can be regarded as the length of the beam. Hence,

$$s = \frac{l^2}{2\,\rho}. \qquad [126]$$

Substituting for ρ from equation 67

$$s = \frac{l^2\,\mathrm{S}}{2\,w\,\mathrm{E}}. \qquad [127]$$

FIG. 459.



If the section is symmetrical with reference to the neutral axis, as in the case represented in Fig. 454, so that $2\,w = h$, and further if δ be written for $\frac{\mathrm{S}}{\mathrm{E}}$ (see equation I., page 282),

$$s = \delta\,\frac{l^2}{h}. \qquad [128]$$

In a beam supported at both ends and loaded in the centre, the deflection curve will evidently be a circle, if each half of the beam be of the form shown either in Figs. 455 and 456, or in Figs. 457 and 458, the thin end being placed at the abutment. In the calculations connected with Fig. 401, it was assumed that the deflection curve was a circle. This assumption therefore requires that the beam should have either of the forms indicated above when its cross-section is rectangular.

Since $\frac{I}{M}$ is constant it can be replaced by its value at the point of fixing, viz. $\frac{I_1}{K l}$, and equation 121 then becomes

$$\rho = \frac{E I_1}{K l};$$

whence, from equation 126,

$$s = \frac{K l^2}{2 E I_1}. \qquad [128A]$$

Hence, in a beam of the form shown in Figs. 457 and 458, in which the curvature is constant, the deflection is 1·5 times greater than that of a prismatic beam of the same depth see equation 79).

§ 52.—Resistance of Long Columns to Bending and Buckling.

If the straight prismatic beam shown in Fig. 460 be subject to forces K K producing compression, the points of application being at the centre of gravity of the end sections, and the forces acting in the direction of the length of the beam, the stress will be uniformly distributed over the area of every cross-section of the beam, and if F is the area of the cross-section the stress per unit of area will be

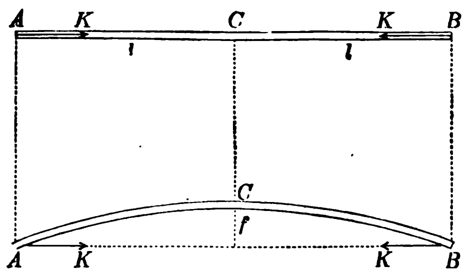
$$S_1 = \frac{K}{F}. \qquad [129]$$

Let it be supposed that by any means whatever the column is bent until the height of arc of the curve formed is f (Fig. 461), and further that the bent column is acted upon by

the two forces K, and that these forces are of themselves able to maintain the column in this bent condition.

The stress at any cross-section can in this case be considered as made up of two parts: the first is the uniformly distributed compression S_1 (from equation 129), and the second is the

FIGS. 460 AND 461.



bending stress. The fibres on the concave side will evidently be compressed by the bending, and since the maximum bending moment

$$M = Kf \qquad [130]$$

occurs at C, the greatest compression will also be at this point. Substituting for M its value from equation 52, the greatest intensity of compression S_2 due to the bending, is

$$S_2 = \frac{w}{I} Kf. \qquad [131]$$

And the greatest compression per unit of area in the column is evidently

$$S = S_1 + S_2. \qquad [132]$$

From the above it follows that there are two different ways in which a column can resist the action of a compressive force K. The first is illustrated by Fig. 460, and in this case the compression is uniformly distributed over the section and equal to S_1 per unit of area. The second way is shown in Fig. 461, and in this case the maximum compression attains the greater value $S_1 + S_2$ per unit of area. In determining the safe

section F of the column, it becomes a question whether the column resists the force K in the first or in the second manner. If it be in the first way F can be found directly from equation 129, by substituting for S_1 the safe resistance to crushing; but if it be in the second way the section must be such that the sum $S_1 + S_2$ is equal to the safe resistance to crushing.

In the case of long thin columns the smallest accidental curvature is sufficient to enable the compressive force to produce bending. Therefore in such a case the second mode of resistance obtains (unless the column be so supported along its length as to preclude the possibility of its bending), and therefore to the direct compression S_1 per unit of area must be added the compression S_2 due to bending, giving to the lever arm f its greatest possible value.

Now a value can be assigned to f of which it may be said with certainty that if the column is sufficiently strong to resist safely the force K, the height of the arc to which the column may be bent will not reach f. For let it be supposed that the curvature at every point of the column is the same; that is, the column will be bent into the arc of a circle. Now if the bending be so great that the greatest compression due to it alone is equal to the elastic limit, it is evident that by adding the direct compression the elastic limit will be overstepped. If the force K could produce such a state of things the column must be considered too weak. If therefore the corresponding value of f be substituted in equation 131 the value of S_2 found will obviously be greater than the compression due to the bending produced by the force K would be in a column of sufficient strength, and consequently the section F found by using this value of S_2 will be rather greater than required.

If in equation I. (§ 44) s means the elastic limit of compression, δ will be the shortening per unit of length at the elastic limit, and the sought value of f can therefore be found from equation 127 by substituting δ for $\frac{S}{E}$ and f for s, thus:

$$f = \frac{l^2 \delta}{2 w}. \qquad [133]$$

Then if in equation 131 f be replaced by this value, and K by the value obtained from equation 129,

$$S_2 = \frac{\delta l^2 F}{2 I} \cdot S_1. \qquad [134]$$

Whence the greatest compression in the column is (equation 132)

$$S = S_1\left(1 + \frac{\delta}{2} \cdot \frac{F l^2}{I}\right); \qquad [135]$$

or putting the whole length of the column $2\,l = L$ and representing the ratio $\frac{S}{S_1}$ by n,

$$n = \frac{S}{S_1} = 1 + \frac{\delta}{8} \cdot \frac{F L^2}{I}. \qquad [136]$$

In this equation S_1 is the uniform compression per unit of area that would occur were it not for the bending, and n is the number of times the greatest intensity of compression S may be made to exceed S_1 by the bending—or in other words, S is the safe stress per unit of area of the cross-section that can be applied to the long column under consideration.

The number n *therefore gives the fraction of the safe resistance to crushing that can be applied to a long column, so that it may be safe as regards bending.*

Substituting the value of δ at the elastic limit for various materials, the following formulæ are obtained:

Cast iron	$\delta = \frac{15}{10000}$	$n = 1 + \frac{1}{5333}\frac{F L^2}{I}$.	[137]
Wrought iron	$\delta = \frac{15}{20000}$	$n = 1 + \frac{1}{10666}\frac{F L^2}{I}$.	[138]
Wood	$\delta = \frac{1 \cdot 8}{1000}$	$n = 1 + \frac{1}{4444}\frac{F L^2}{I}$.	[139]

In these equations I is the moment of inertia of the section of the column about that axis through the centre of gravity of the section, which is perpendicular to the plane of bending, that is, at right angles to the direction in which bending takes place easiest. It is in fact the minimum bending moment of the section.

If, for instance, the section is *rectangular*, H the greater and B the smaller side, $I = \frac{H B^3}{12}$; if, however, H is the smaller of the two dimensions, $I = \frac{B H^3}{12}$. In the last case, therefore,

$$\frac{F}{I} = \frac{B H}{\frac{1}{12} B H^3} = \frac{12}{H^2},$$

and the above formulæ become

$$\text{Cast iron,} \qquad n = 1 + 0 \cdot 00225 \left(\frac{L}{H}\right)^2; \qquad [140]$$

$$\text{Wrought iron,} \qquad n = 1 + 0 \cdot 001125 \left(\frac{L}{H}\right)^2; \qquad [141]$$

$$\text{Wood,} \qquad n = 1 + 0 \cdot 0027 \left(\frac{L}{H}\right)^2; \qquad [142]$$

from which the following table has been constructed:

	$\frac{L}{H} =$	10	20	30	40	50
Cast iron,	$n =$	1·225	1·9	3·025	4·6	6·625
Wrought iron,	$n =$	1·1125	1·45	2·0125	2·8	3·8125
Wood,	$n =$	1·27	2·08	3·43	5·32	7·75

Thus if 6 kilos. per square millimetre be taken as the safe resistance to crushing of wrought iron, then the safe compression per square millimetre on a long column of the same material of rectangular section whose length is twenty times its least dimension H, is

$$S_1 = \frac{S}{n} = \frac{6}{1 \cdot 45} = 4 \cdot 14 \text{ kilos.};$$

and if for instance $H = 10$ millimetres, and $B = 40$ millimetres, the greatest safe load that could be placed on it is,

$$K = F S_1 = 400 \times 4 \cdot 14 = 1656 \text{ kilos.}$$

If the section is a hollow rectangle as in Fig. 427,

$$\frac{F}{I} = \frac{B H - b h}{\frac{1}{12} B H^3 - \frac{1}{12} b h^3},$$

and the formula for wrought iron becomes,

$$n = 1 + 0 \cdot 001125 \frac{(B H - b h) L^2}{B H^3 - b h^3}. \qquad [143]$$

Thus for a rectangular wrought iron tube the thickness of which is $\frac{1}{20}$th of the exterior dimensions,

$$n = 1 + 0 \cdot 00062 \left(\frac{L}{H}\right)^2; \qquad [144]$$

and if at the same time H is $\frac{1}{20}$th of the length,

$$n = 1 \cdot 248.$$

Such a tube could therefore only be loaded with $S_1 = \frac{6}{1 \cdot 248} = 4 \cdot 8$ kilos. per square millimetre.

For a circular tube of exterior diameter D and interior diameter d,

$$\frac{F}{I} = \frac{\frac{\pi}{4}(D^2 - d^2)}{\frac{\pi}{64}(D^4 - d^4)} = \frac{16}{D^2 + d^2},$$

and equation 138 becomes in the case of wrought iron,

$$n = 1 + 0 \cdot 0015 \left(\frac{L^2}{D^2 + d^2}\right), \qquad [145]$$

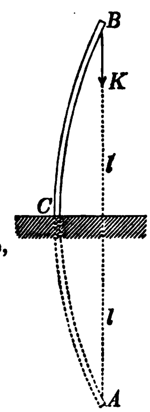
Let $\frac{d}{D} = 0 \cdot 9$, $\frac{L}{D} = 20$, then $n = 1 \cdot 3315$ and $S_1 = 4 \cdot 5$ kilos.

If, however, $d = 0$, and $\frac{L}{D} = 20$, it will be found that $n = 1 \cdot 6$, and $S_1 = 3 \cdot 75$ kilos.

If it be supposed that one half of the column of Fig. 461 is firmly fixed (encastré), the state of stress of the other half —and consequently the greatest compression at C—will not be altered thereby. Therefore the general equation 136 is also true in the case represented in Fig. 462.* And writing $2l$ instead of L,

$$n = 1 + \frac{\delta}{2} \cdot \frac{F l^2}{I}. \qquad [146]$$

FIG. 462.



If a round wrought-iron column be loaded in this manner, and the length B C = l is twenty times the diameter, it will be found that $n = 3 \cdot 4$, and $S_1 = \frac{6}{3 \cdot 4} = 1 \cdot 76$ kilos. For instance, if the area of the section is 100 sq. millim., the safe load is K = 176 kilos. If the same column were placed in the conditions of Fig. 461, it could carry safely a load K = 375 kilos.

It appears from the above that the sectional areas obtained in §§ 41 and 42 for the compression braces, can only be adopted if by their construction the value of n for them differs very little from 1.

* This is the case of a long column having one end fixed and the other free but *not* "guided." If the free end were guided the strength of the column would be materially increased. This subject will be found very fully treated in 'Der Constructeur,' by Prof. Reuleaux.—TRANS.

It also appears that, as a rule, a greater section is required to resist compression than to resist the same amount of tension, and that the greater the ratio of the length to the least dimension of the cross-section, the greater must be the section to resist compression, but not proportionately.

And lastly, that if the length remains the same and also the form of section, the less the load to be borne, the greater, proportionately, will the section be.

From this last remark it follows that, if possible, bars in compression should not be split up into several isolated parts. In this respect, therefore, the simplest forms of construction are the best; for instance, braced girders with a single triangulation are to be preferred to trellis girders. Trellis girders have, however, this advantage (already pointed out in § 43), that owing to the greater number of points of support obtained, there is a great saving of material in the longitudinal girders. When deciding on the depth of a girder, it should be remembered that the resistance of the compression braces diminishes rapidly as the depth increases.

Further, it appears that the design of the structure should be such that the compression braces are as short as possible. For this reason, girders in which the verticals are in compression and the diagonals in tension, are generally to be preferred to other forms.

No general rule can, however, be framed by means of which it can be decided what form, what number of triangulations, and what depth a girder should have in order that the bridge may contain the least quantity of material.

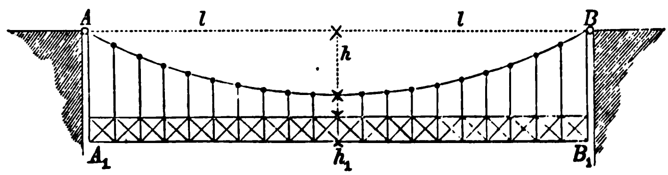
SIXTEENTH CHAPTER.

§ 53.—COMPOUND LATTICE AND SUSPENSION BRIDGE—SPAN 60 METRES.

Determination of the best ratio between the depth of the girders and the height of the arc of the suspension chains.

THE general design of the bridge is shown in Fig. 463. In this figure, however, the points A and B are represented as fixed points, so as to remove at first from the calculations the consideration of the back-stays. The arrangement of the bridge in this respect is shown in Fig. 491.* The material is wrought iron.

FIG. 463.



The two lattice girders are continuous between the abutments, and the section of the booms is the same throughout. Each girder is connected to a suspension chain, by means of vertical rods attached to each top joint of the girder and to a point in the chain. The suspension chains are in the form of a parabola, or, more strictly speaking, of a polygon inscribed in a parabola.

The total load on the bridge is 0·575 kilo. per millimetre run, consisting of a permanent load $p = 0 \cdot 375$ kilo., and a moving load $m = 0 \cdot 2$ kilo. per millimetre run.

The first step is to decide what the proportion between the depth of the girder and the height of arc of the chain should

* This bridge was constructed by a German firm to be sent out to the Brazils, and the author, at the request of the manufacturer, furnished the following calculations.

be, in order that the quantity of material in the structure may be a minimum. Although the greatest stress will be reached in certain parts of the girder when the moving load is unevenly distributed, yet for the present inquiry this can be ignored, and the moving load considered as covering the bridge.

When the temperature varies, the length of the suspension chains will alter, and consequently the distribution of the load between the two systems will also alter. The changes of temperature must therefore be taken into account.

If δ is the greatest elongation per unit of length produced in the chains by the load alone, the deflection of their lowest points will be, according to § 45,

$$\sigma = \tfrac{3}{4}\,\delta\,\frac{l^2}{h}\left(1 + \tfrac{2}{3}\,\frac{h^2}{l^2}\right). \qquad [147]$$

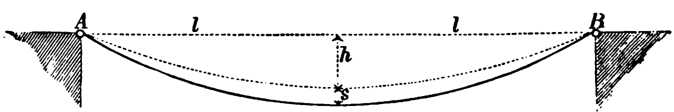
In the present case, as will be seen, the ratio $\frac{h}{l}$ is small; therefore

$$\sigma = \tfrac{3}{4}\,\delta\,\frac{l^2}{h}\ \text{(approximately)}. \qquad [148]$$

And if, further, Δ is the elongation per unit of length due to the greatest increase of temperature (Fig. 464), the deflection due to both causes is

$$s = \tfrac{3}{4}\,(\delta + \Delta)\,\frac{l^2}{h}. \qquad [149]$$

FIG. 464.



The deflection in the centre of a prismatic beam, subject to a uniformly distributed load, is from equation 99, § 51,

$$s_1 = \tfrac{5}{24}\,\frac{q\,l^4}{E_1\,I_1}, \qquad [150]$$

in which equation q is the load per unit of length, I_1 the moment of inertia of the section (supposed symmetrical about the neutral axis), and E_1 is the modulus of elasticity of the material. (Fig. 465.)

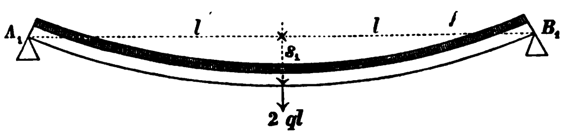
The greatest bending stress, S_1, in the beam can be found from equation 64, § 50,

$$\frac{S_1}{\frac{1}{2} h_1} I_1 = \frac{q l^2}{2}. \qquad [151]$$

Substituting the value of I_1, obtained from this equation in equation 150,

$$s_1 = \frac{5}{6} \frac{S_1 \, l^2}{E_1 \, h_1}; \qquad [152]$$

FIG. 465.



and writing δ_1, the greatest elongation per unit of length occurring in the beam instead of $\frac{S_1}{E_1}$,

$$s_1 = \frac{5}{6} \delta_1 \frac{l^2}{h^1}. \qquad [153]$$

According to § 49, the economical ratio $\frac{h_1}{h}$ is found by equating the two deflections, thus:—

$$\frac{3}{4} (\delta + \Delta) \frac{l^2}{h} = \frac{5}{6} \delta_1 \frac{l^2}{h_1}. \qquad [154]$$

Whence

$$\frac{h_1}{h} = \frac{\frac{5}{6} \delta_1}{\frac{3}{4} (\delta + 4)}. \qquad [155]$$

If it be supposed that the bridge is erected at the mean temperature, Δ will depend on the difference between the greatest temperature that occurs and the mean temperature. Since the chains are of wrought iron, if this difference of temperature is 41° C., $\Delta = \frac{1}{2000}$; but if the difference is 20°·5 C., $\Delta = \frac{1}{4000}$. Further, the modulus of elasticity of wrought iron is 20,000 kilos. per square millimetre, and the safe stress can be taken at from 5 to 10 kilos. per square millimetre; therefore δ and δ_1 vary between $\frac{5}{20000}$ and $\frac{10}{20000}$. Assuming

that the height of the arc of the chains $h = 4$ metres, the following table can be formed from equation 155 :—

δ (Chain).	δ_1 (Girder).	Δ (Temperature-elongation).	$\frac{h_1}{h}$.	h_1.
				metres.
$\frac{6}{20000}$	$\frac{6}{20000}$	$\frac{1}{4000}$ (20°·5 C.)	$\frac{20}{33}$	2·424
		$\frac{1}{2000}$ (41° C.)	$\frac{5}{12}$	1·667
$\frac{8}{20000}$	$\frac{6}{20000}$	$\frac{1}{4000}$ (20°·5 C.)	$\frac{20}{39}$	2·051
		$\frac{1}{2000}$ (41° C.)	$\frac{10}{27}$	1·48
$\frac{8}{20000}$	$\frac{5}{20000}$	$\frac{1}{4000}$ (20°·5 C.)	$\frac{50}{117}$	1·71
		$\frac{1}{2000}$ (41° C.)	$\frac{25}{81}$	1·234
$\frac{10}{20000}$	$\frac{5}{20000}$	$\frac{1}{4000}$ (20°·5 C.)	$\frac{10}{27}$	1·48
		$\frac{1}{2000}$ (41° C.)	$\frac{5}{18}$	1·111

In using this table it must be remembered that, although both the chains and the girders are made of wrought iron, yet, for the following reasons, the value of δ_1 should be taken smaller than δ. 1. δ_1 in the girders depends on the resistance to compression, whereas in the chains δ depends only on the resistance to tension. 2. Because even when the bridge is fully loaded and the temperature is at its highest, the maximum stresses in the girders are not reached; for they are also subject to the effect of the horizontal pressure of the wind, and of the unequal distribution of the moving load. 3. Because the points of attachment of the chains, although considered fixed, really approach each other slightly owing to the extension of the land ties, and this has the same effect as if the elasticity of the chains were increased.

It would appear, therefore, that if $h = 4$ metres, then $h_1 = 1 \cdot 5$ metre is a good value for the depth of the girders, this value being the arithmetic mean of those in last column, when the two first are omitted.

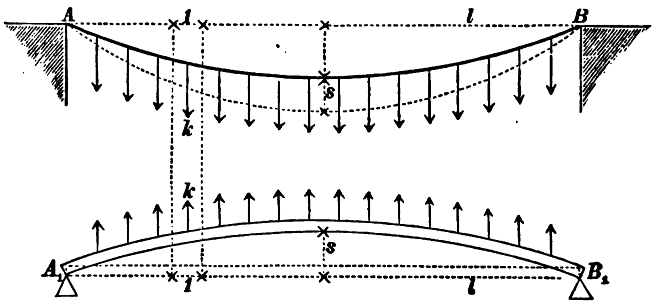
§ 54.—Calculation of the Stresses produced by Changes of Temperature.

When the temperature diminishes, the chains shorten and their lowest points consequently rise. This induces stresses in the chains and girders, which are to be added to those produced

by the loads; but they will be calculated separately by considering that the structure is totally unloaded.

The ends, A_1 and B_1, of the girders can be considered as capable of resisting an upward as well as a downward reaction (see § 66). The shortening of the chains due to a diminution of temperature will be accompanied by an upward bending of the girders. (Fig. 466.) The resistance thus pro-

FIG. 466.



duced will have the same effect on the chains as a uniformly distributed load—say, k per unit of length; and the chains will be prevented from rising to the same height that they otherwise would do. The actual amount s the lowest points of the chains rise, is therefore equal to the difference between the upward deflection s_1, produced by the diminution of temperature, and the deflection s_2, due to the load k per unit of length.

If the shortening per unit of length due to a decrease in temperature be represented by Δ,

$$s_1 = \tfrac{3}{4}\,\Delta\,\frac{l^2}{h}. \qquad [156]$$

The horizontal tension in the chains, due to the load k per unit of length, is, according to § 8,

$$H = \frac{k\,l^2}{2\,h}. \qquad [157]$$

If, therefore, E is the modulus of elasticity of the material of the chains, and F the sectional area of both chains at their lowest point, then δ the elongation due to k is

$$\delta = \frac{H}{E\,F} = \frac{k\,l^2}{2\,E\,F\,h}, \qquad [158]$$

whence the deflection of the lowest points of chains is

$$s_2 = \tfrac{3}{4}\,\delta\,\frac{l^2}{h} = \tfrac{3}{8}\,\frac{k\,l^4}{\mathrm{E\,F}\,h^2}, \qquad [159]$$

and the actual upward deflection is therefore

$$s = s_1 - s_2 = \tfrac{3}{4}\,\Delta\,\frac{l^2}{h} - \tfrac{3}{8}\,\frac{k\,l^4}{\mathrm{E\,F}\,h^2}. \qquad [160]$$

The upward deflection of the girders must be equal to s, and since it is produced by an upward uniform load k per unit of length,

$$s = \tfrac{5}{24}\,\frac{k\,l^4}{\mathrm{E_1\,I_1}}. \qquad [161]$$

Equating the two values found for s,

$$\tfrac{3}{4}\,\Delta\,\frac{l^2}{h} - \tfrac{3}{8}\,\frac{k\,l^4}{\mathrm{E\,F}\,h^2} = \tfrac{5}{24}\,\frac{k\,l^4}{\mathrm{E_1\,I_1}}; \qquad [162]$$

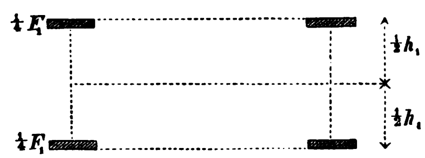
or

$$k = \frac{\tfrac{3}{4}\,\dfrac{\Delta}{h}}{\tfrac{3}{8}\,\dfrac{l^2}{\mathrm{E\,F}\,h^2} + \tfrac{5}{24}\,\dfrac{l^2}{\mathrm{E_1\,I_1}}}. \qquad [163]$$

If $\mathrm{F_1}$ is the sum of the effective sectional areas of the four booms of the two lattice girders (Fig. 467),

$$\mathrm{I_1} = 4\,(\tfrac{1}{4}\,\mathrm{F_1})\,(\tfrac{1}{2}\,h_1)^2 = \frac{\mathrm{F_1}\,h_1^2}{4}. \qquad [164]$$

FIG. 467.



And substituting in the above equation

$$k = \frac{2\,\Delta\,\mathrm{E\,F}\,h}{l^2\left(1 + \frac{20}{9}\cdot\frac{\mathrm{E}}{\mathrm{E_1}}\cdot\frac{\mathrm{F}}{\mathrm{F_1}}\cdot\frac{h^2}{h_1^2}\right)}. \qquad [165]$$

For example, let $\Delta = \frac{1}{2000}$, $E = E_1 = 20{,}000$ kilos., $h = 4000$ mm., $F = 7500$ sq. mm., $F_1 = 15{,}000$ sq. mm., $l = 30{,}000$ mm.; then $k = 0\cdot074{,}896$ kilo. per millimetre, or nearly 75 kilos. per metre run.

If Δ were negative, k would also be negative; and thus it is seen that the action of an increase of temperature is to unload the chains and load the girders by the amount k per unit of length. Thus, if the bridge be constructed at the mean temperature, the girders will be unloaded by the amount of 75 kilos. per metre run, and the chains loaded by the same amount when the temperature is 41° C. below the mean; and when the temperature is 41° C. above the mean, exactly the reverse will take place. Therefore the load k, which can be called a *temperature load*, is applied to the chains when the temperature decreases, and to the girders when the temperature increases and produces stresses which must be added to those due to the ordinary loading. In the chains this temperature stress is by equation 157

$$S = \frac{k\,l^2}{2\,F\,h} = 1\cdot1234 \text{ kilos. per sq. millimetre}; \qquad [166]$$

and for the booms of the girders, from equations 151 and 164,

$$S_1 = \frac{k\,l^2}{F_1\,h_1} = 2\cdot996 \text{ kilos. per sq. millimetre.} \qquad [167]$$

§ 55.—Calculation of the Stresses produced by the Permanent Load.

Let the uniformly distributed load on the bridge be p per unit of length, and let $n\,p$ be the portion carried by the chains: $(1-n)\,p$ will therefore be the load on the girders (Figs. 468, 469, and 470). Now, similarly to equation 159, the deflection of the lowest points of the chains is

$$s = \tfrac{3}{8}\,\frac{n\,p\,l^4}{E\,F\,h^2}, \qquad [168]$$

and the deflection of the girders at the centre is (equations 161 and 164)

$$s = \tfrac{1}{6}\,\frac{(1-n)\,p\,l^4}{E_1\,F_1\,h^2_1}. \qquad [169]$$

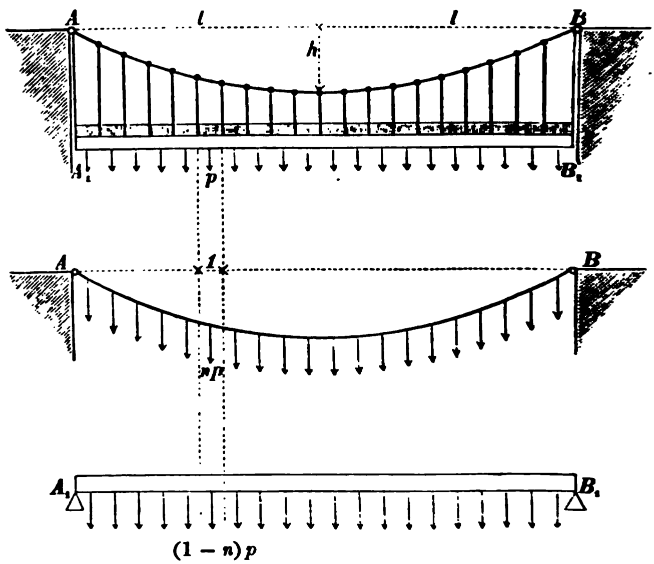
Therefore, equating these two values of s,

$$\frac{3}{8}\frac{n\,p\,l^4}{E\,F\,h^2} = \frac{5}{6}\frac{(1-n)\,p\,l^4}{E_1\,F_1\,h^2_1}, \qquad [170]$$

or

$$n = \frac{1}{1+\frac{9}{20}\cdot\frac{E_1}{E}\cdot\frac{F_1}{F}\cdot\frac{h^2_1}{h^2}}. \qquad [171]$$

FIGS. 468, 469, AND 470.



As before, let $\frac{E_1}{E} = 1$, $\frac{F_1}{F} = 2$, $\frac{h_1}{h} = \frac{1 \cdot 5}{4}$; then

$$n = \frac{1}{1+\frac{81}{640}} = 0 \cdot 887656. \qquad [172]$$

If, therefore, the dead load per metre run is 375 kilos. (or $p = 0 \cdot 375$ per millimetre run), it is distributed as follows:

On the chains,

$$n\,p = 0 \cdot 33287 \text{ kilo. per millimetre run;}$$
$$= 332 \cdot 87 \text{ kilos. per metre run.} \qquad [173]$$

On the girders,

$$(1-n)\,p = 0 \cdot 04213 \text{ kilo. per millimetre run;}$$
$$= 42 \cdot 13 \text{ kilos. per metre run.} \qquad [174]$$

And the stresses produced in the chains and in the booms of the girders are respectively,

$$S = \frac{n\,p\,l^2}{2\,F\,h} = 4 \cdot 993 \text{ kilos. per sq. millimetre;} \qquad [175]$$

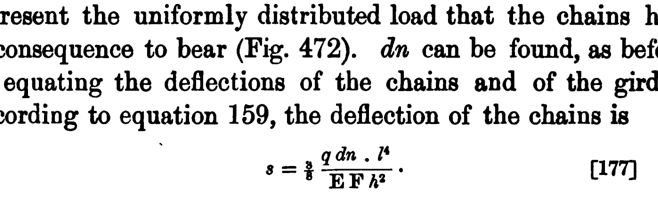
$$S_1 = \frac{(1-n)\,p\,l^2}{F_1\,h_1} = 1 \cdot 685 \text{ kilo. per sq. millimetre.} \qquad [176]$$

§ 56.—Calculation of the Stresses produced by a Moving Load.

It was shown in § 8 that if the curve of equilibrium of a chain is a parabola, the load must be equally distributed over the span or horizontal projection. Now, in the present case the deflection of the girders is but small; it may therefore be assumed that *the chains retain always their parabolic form.* It follows that the chains must in all cases be uniformly loaded, even if the load be concentrated at one or more points on the girders.

In Fig. 471 let the elements dz of the span, at equal distances z from the centre, be loaded with q per unit of length; then $q\,dz$ will be the load on each element. Further, let $q\,dn$

Fig. 471.



represent the uniformly distributed load that the chains have in consequence to bear (Fig. 472). dn can be found, as before, by equating the deflections of the chains and of the girders. According to equation 159, the deflection of the chains is

$$s = \tfrac{3}{8}\,\frac{q\,dn\,.\,l^4}{E\,F\,h^2}. \qquad [177]$$

The deflection of the girders is equal to the deflection that the two loads $q\,dz$, would produce of themselves (Fig. 473) minus

the deflection due to the upward uniformly distributed load $q\,dn$ per unit of length (Fig. 474).

From equation 97 the first part is

$$s_1 = \frac{q\,dz}{6\,E_1\,I_1}(2\,l^3 - 3\,l\,z^2 + z^3); \qquad [178]$$

and from equation 99 the second part is

$$s_2 = \tfrac{5}{24}\,\frac{q\,dn\,.\,l^4}{E_1\,I_1}. \qquad [179]$$

FIG. 472.

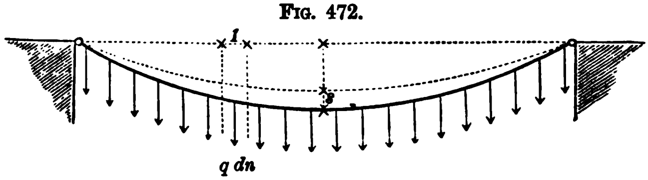


FIG. 473.

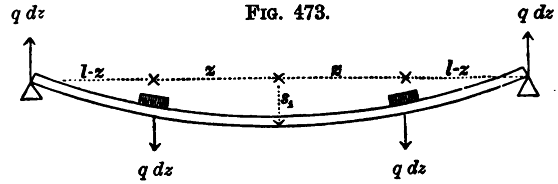
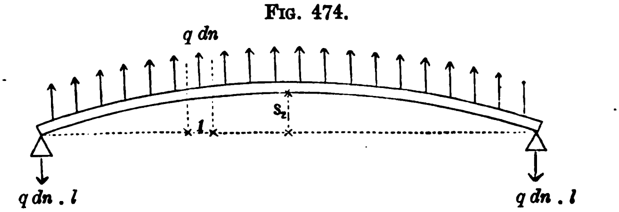


FIG. 474.



Hence the actual deflection is

$$s = s_1 - s_2 = \frac{q\,dz}{6\,E_1\,I_1}(2\,l^3 - 3\,l\,z^2 + z^3) - \tfrac{5}{24}\,\frac{q\,dn\,.\,l^4}{E_1\,I_1}; \qquad [180]$$

and equating the two values found for s,

$$\tfrac{3}{8}\,\frac{q\,dn\,.\,l^4}{E\,F\,h^2} = \frac{q\,dz}{6\,E_1\,I_1}(2\,l^3 - 3\,l\,z^2 + z^3) - \tfrac{5}{24}\,\frac{q\,dn\,.\,l^4}{E_1\,I_1}; \qquad [181]$$

or,

$$dn = \frac{(2\,l^3 - 3\,l\,z^2 + z^3)\,dz}{\tfrac{5}{4}\,l^4\left(1 + \tfrac{9}{5}\,\frac{E_1I_1}{E\,F\,h^2}\right)}. \qquad [182]$$

Substituting from equation 164

$$I_1 = \frac{F_1 h_1^2}{4},$$

and integrating the right-hand side between the limits z_1 and z_2, and the left-hand side between the limits n_1 and n_2,

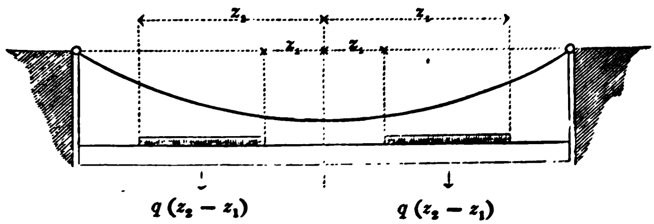
$$\int_{n_1}^{n_2} dn = \int_{z_1}^{z_2} \frac{(2l^3 - 3lz^2 + z^3)}{\frac{5}{4}l^4\left(1 + \frac{9}{20}\cdot\frac{E_1}{E}\cdot\frac{F_1}{F}\cdot\frac{h_1^2}{h^2}\right)}\, dz; \qquad [188]$$

or,

$$n_2 - n_1 = \frac{2l^3(z_2 - z_1) - l(z_2^3 - z_1^3) + \frac{1}{4}(z_2^4 - z_1^4)}{\frac{5}{4}l^4\left(1 + \frac{9}{20}\cdot\frac{E_1}{E}\cdot\frac{F_1}{F}\cdot\frac{h_1^2}{h^2}\right)}. \qquad [184]$$

Thus, when the bridge is loaded as shown in Fig. 475, the uniformly distributed load the chains have to carry is $(n_2 - n_1)\,q$ per unit of length of the span.

FIG. 475.



(Putting $z_1 = 0$, $z_2 = l$, and $n_1 = 0$, $n_2 = n$, so that the uniformly distributed load covers the bridge,

$$n = \frac{1}{1 + \frac{9}{20}\cdot\frac{E_1}{E}\cdot\frac{F_1}{F}\cdot\frac{h_1^2}{h^2}} = [illegible] = 0{\cdot}887656; \qquad [185]$$

or the same value that was obtained from equation 172.)

Evidently each of the loads $q\,(z_2 - z_1)$ has the same effect on the chains; each of them, therefore, produces a load of $\left(\frac{n_2 - n_1}{2}\right) q$ per unit length of the span. This is shown in Figs. 476 and 477. In this case the deflection of the girders is compounded of the downward deflection due to the

load $q\ (z_2 - z_1)$, minus the upward deflection due to the uniformly distributed upward load $\left(\frac{n_2 - n_1}{2}\right) q$ produced by the chains. (Figs. 478 and 479.)

FIG. 476.

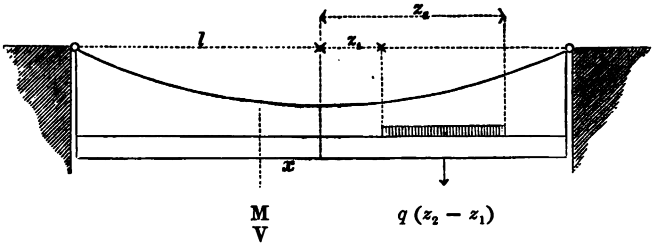


FIG. 477.

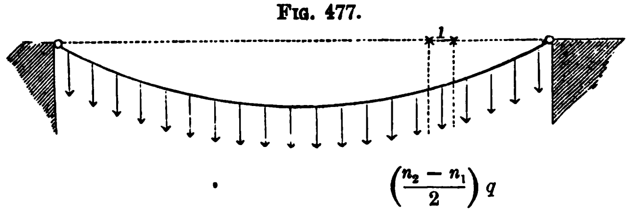


FIG. 478.

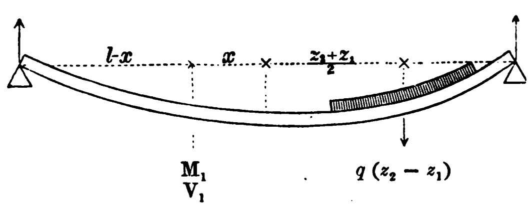
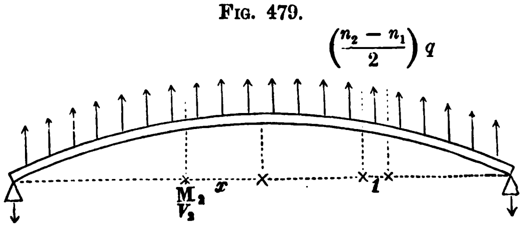


FIG. 479.



Therefore the bending moment at a point distant x to

the left of the centre, is the difference between the bending moment due to the load $q\ (z_2 - z_1)$ Fig. 478,

$$M_1 = D\,(l - x); \qquad [186]$$

and the bending moment due to the loading shown in Fig. 479,

$$M_2 = \left(\frac{n_2 - n_1}{2}\right) q \left(\frac{l^2 - x^2}{2}\right). \qquad [187]$$

The resulting bending moment is therefore

$$M = M_1 - M_2 = D\,(l - x) - \left(\frac{n_2 - n_1}{2}\right) q \left(\frac{l^2 - x^2}{2}\right). \qquad [188]$$

Similarly, it will be seen that the shearing force at the same point is

$$V = V_1 - V_2 = D - \left(\frac{n_2 - n_1}{2}\right) q\,x. \qquad [189]$$

In both these equations, D is to be replaced by its value derived from Fig. 478, viz.:

$$D = \frac{q\,(z_2 - z_1)}{2}\left(1 - \frac{z_2 + z_1}{2\,l}\right). \qquad [190]$$

It will be observed that the direction of M and V, when the bridge is fully loaded, has been taken as the positive direction.

§ 57.—Determination of the Worst Condition of Loading for the Girders.

To find what conditions of loading produce the greatest bending moment M, and the greatest shearing force V, respectively at any given section of the girders, the points must first be found where a load must be placed so that M = 0 and V = 0 respectively; for it is evident that these points separate the loads that produce positive from those that produce negative values of M and V respectively.

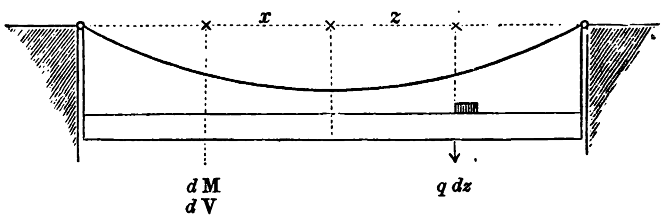
The values of M and V found in equations 188 and 189 can be regarded as the sum of the increments due to each element

of the load $q\,(z_2 - z_1)$. Let $d\,\mathrm{M}$ and $d\,\mathrm{V}$ represent the increments produced by a load $q\,d\,z$ situated at a distance z from the centre (Fig. 480). Then proceeding as in the former case (equations 188 and 189), the following equations are obtained:

$$d\,\mathrm{M} = \frac{q\,d\,z\,(l - z)\,(l - x)}{2\,l} - \frac{q\,d\,n\,(l^2 - x^2)}{4}, \qquad [191]$$

$$d\,\mathrm{V} = \frac{q\,d\,z\,(l - z)}{2\,l} - \frac{q\,d\,n}{2}\,x; \qquad [192]$$

Fig. 480.



which become, when $d\,n$ is replaced by its value obtained from equation 182

$$d\,\mathrm{M} = \frac{q\,d\,z\,(l - z)\,(l - x)}{2\,l}\left\{1 - \frac{2\,(l + x)\,(2\,l^2 + 2\,l\,z - z^2)}{l^3\left(5 + 9\,\dfrac{\mathrm{E}_1\,\mathrm{I}_1}{\mathrm{E\,F}\,h^2}\right)}\right\} \qquad [193]$$

$$d\,\mathrm{V} = \frac{q\,d\,z\,(l - z)}{2\,l}\left\{1 - \frac{4\,x\,(2\,l^2 + 2\,l\,z - z^2)}{l^3\left(5 + 9\,\dfrac{\mathrm{E}_1\,\mathrm{I}_1}{\mathrm{E\,F}\,h^2}\right)}\right\}. \qquad [194]$$

Evidently the position of the load which produces no bending moment at the section under consideration, can be found by putting $d\,\mathrm{M} = 0$. Thus (writing u instead of z as a distinction):

$$\frac{l^3\left(5 + 9\,\dfrac{\mathrm{E}_1\,\mathrm{I}_1}{\mathrm{E\,F}\,h^2}\right)}{2\,(l + x)} = 2\,l^2 + 2\,l\,u - u^2. \qquad [195]$$

substituting $\mathrm{I}_1 = \dfrac{\mathrm{F}_1 h_1^{\,2}}{4}$ and solving,

$$\frac{u}{l} = 1 - \sqrt{\frac{6\,x + l\left(1 - \frac{9}{4}\,\dfrac{\mathrm{E}_1}{\mathrm{E}} \cdot \dfrac{\mathrm{F}_1}{\mathrm{F}} \cdot \dfrac{h_1^{\,2}}{h^2}\right)}{2\,(l + x)}}. \qquad [196]$$

Only one root of this quadratic equation applies to the question under consideration; for the other root would make $\frac{u}{l} > 1$, which is evidently inadmissible in the present case. The same remark refers to equation 197 below.

Proceeding in the same manner for $d\,\mathrm{V}$, and representing by v the value of z, obtained by putting $d\,\mathrm{V} = 0$

$$\frac{l^3}{4\,x}\left(5 + 9\,\frac{\mathrm{E}_1\,\mathrm{I}_1}{\mathrm{E\,F}\,h^2}\right) = 2l^2 + 2\,l\,v - v^2; \qquad [197]$$

or,

$$\frac{v}{l} = 1 - \sqrt{3 - \frac{l}{4\,x}\left(5 + \frac{9}{4}\,\frac{\mathrm{E}_1}{\mathrm{E}}\cdot\frac{\mathrm{F}_1}{\mathrm{F}}\cdot\frac{h_1^2}{h^2}\right)}. \qquad [198]$$

As before, put $\frac{\mathrm{E}_1}{\mathrm{E}} = 1$, $\frac{\mathrm{F}_1}{\mathrm{F}} = 2$, $\frac{h_1}{h} = \frac{1 \cdot 5}{4}$; then the equations for u and v become

$$\frac{u}{l} = 1 - \sqrt{\frac{768\,x + 47\,l}{256\,(l + x)}}, \qquad [199]$$

$$\frac{v}{l} = 1 - \sqrt{3 - \frac{721\,l}{512\,x}}. \qquad [200]$$

Both these equations refer to the case when the load is placed to the right of the section under consideration. If u_1 and v_1 are the distances to the left of the centre of the points where a load must be placed to produce no bending moment or shearing stress respectively at a section situated between them and the centre, it will be found that

$$\frac{u_1}{l} = 1 - \sqrt{\frac{47\,l - 768\,x}{256\,(l - x)}}, \qquad [201]$$

$$\frac{v_1}{l} = 1 - \sqrt{3 + \frac{721\,l}{512\,x}}. \qquad [202]$$

The following table has been computed from the above four equations:—

$\frac{x}{l}$	$\frac{u}{l}$	$\frac{u_1}{l}$	$\frac{v}{l}$	$\frac{v_1}{l}$
− 1	..	− 0·2616	..	− 0·2616
− 0·75	..	− 0·18	..	− 0·0594
− 0·7041	..	..	..	0
− 0·5	..	− 0·0594	..	+ 0·5715
− 0·4694	..	..	..	+ 1
− 0·4082	..	0		
− 0·25	..	+ 0·1358		
− 0·24	..	+ 0·1464		
− 0·2	..	+ 0·1919		
− 0·125	..	+ 0·29535		
− 0·0612	+ 1	+ 0·4118		
0	+ 0·5715	+ 0·5715		
+ 0·0612	+ 0·4118	+ 1		
+ 0·125	+ 0·29535			
+ 0·2	+ 0·1919			
+ 0·24	+ 0·1464			
+ 0·25	+ 0·1358			
+ 0·4082	0			
+ 0·4694	..	..	+ 1	
+ 0·5	− 0·0594	..	+ 0·5715	
+ 0·7041	..	..	0	
+ 0·75	− 0·18	..	− 0·0594	
+ 1	− 0·2616	..	− 0·2616	

§ 58.—Calculation of the Stresses produced in the Booms of the Girders by the Moving Load.

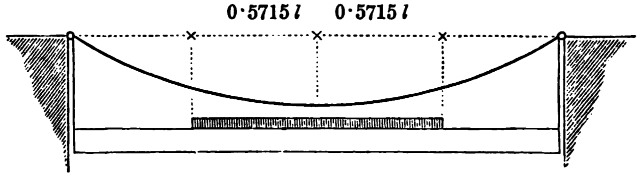
The above table shows that for all values of x between $+ 0\cdot0612\,l$ and $- 0\cdot0612\,l$ there are two zero-points for the bending moment (calling, for shortness, the point where a load must be placed to produce no bending moment, the *zero-point*), and that for all other values of x there is only one zero-point.

When $x = 0$, that is, for the centre of the girders, the two zero-points lie at equal distances, $u = u_1 = 0\cdot5715\,l$, to the right and to the left of the centre. These two zero-points separate those loads that produce positive from those that produce negative bending moments at the centre of the girders. For instance, if the moving load were distributed as shown in Fig. 481, the positive bending moment at the centre would be at its maximum, and its value can be found

as follows:—The first step is to find the part of the load carried by the chains. This can be done by means of equation 183, and in the present case the limits of the integration evidently are $n_1 = 0$ and $n_2 = n$; $z_1 = 0$ and $z_2 = 0 \cdot 5715\, l$. It will then be found that

$$n = 0 \cdot 6983. \qquad [203]$$

FIG. 481.



Now, since the moving load is $m = 0 \cdot 2$ kilo. per millimetre run, in the present case the uniformly distributed load on the chains will be

$$n\,m = 0 \cdot 6983 \times 0 \cdot 2 = 0 \cdot 1396 \text{ kilo. per millimetre run of the span.}$$

This is also the upward uniformly distributed load on the girders.

If M_1 and M_2 denote the bending moments at the centre of the girders due to the loading shown in Figs. 482 and 483 respectively, the resultant bending moment at the centre is

$$M = M_1 - M_2 \qquad [204]$$

and

$$M_1 = m\,z\,l - \frac{m\,z^2}{2}, \qquad [205]$$

$$M_2 = \frac{n\,m\,l^2}{2}. \qquad [206]$$

Further S_m, the stress per unit of area due to the bending moment M, can be found from the equation,

$$M = \frac{S_m I_1}{\frac{1}{2} h_1} = \frac{S_m F_1 h_1}{2},$$

or,

$$S_m = \frac{2\,M}{F_1 h_1}; \qquad [207]$$

and by substituting the above numerical values:

$$S_m = \frac{0 \cdot 0237\, l^2}{F_1 h_1} = 0 \cdot 948 \text{ kilo. per sq. millimetre.} \qquad [208]$$

From equation 176 it will be found that the same stress would be produced by a uniform load on the girders of 23·7 kilos. per metre run; whereas when the moving load covers the bridge the part supported by the girders is only

$$(1 - \cdot 887656) \times 200 = 22 \cdot 5 \text{ kilos. per metre run.}$$

FIG. 482.

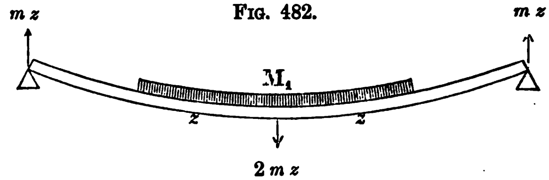
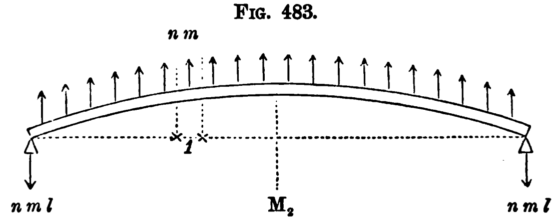
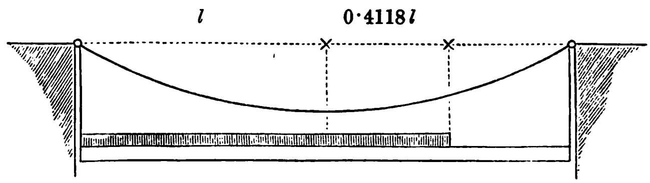


FIG. 483.



As a further example, let it be required to find the maximum bending moment at the section whose distance is $x = 0 \cdot 0612\, l$ to the left of the centre. As will appear from

FIG. 484.



the table, p. 362, the moving load will in this case cover the shaded portion in Fig. 484.

Now, as explained at p. 357, the value of n for the part of the load from the centre to $0 \cdot 4118\, l$ is half the value of n when the load extends to $0 \cdot 4118\, l$ on each side of the centre. The limits in equation 183 can therefore be taken as $n_1 = 0$, $n_2 = 2\, n'$, $z_1 = 0$, and $z_2 = 0 \cdot 4118\, l$; and substituting these values in equation 184, it will be found that

$$n' = \cdot 270172.$$

The same applies to the part of the load from the centre to the left abutment, and from equation 185,

$$n'' = 0 \cdot 443828;$$

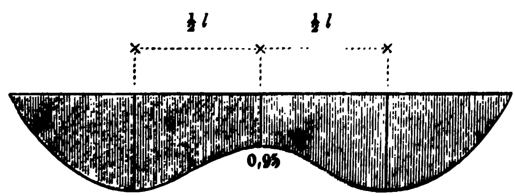
and hence the value of n for the distribution of the moving load shown in Fig. 484 is

$$n = n' + n'' = 0 \cdot 714.$$

Then, proceeding as in the previous case, it will be found that the stress in the booms, at the section under consideration, is:

$$S_m = \frac{0 \cdot 0245\, l^2}{F_1\, h_1} = 0 \cdot 98 \text{ kilo. per sq. millimetre.} \qquad [209]$$

FIG. 485.



The value of S_m for all values of x can be similarly obtained, and the result is expressed graphically in Fig. 485, which shows that the greatest bending moment occurs when $x = \frac{1}{2}\, l$.

§ 59.—Calculation of the Stresses produced by the Permanent and Temperature Loads in the Booms of the Girders.

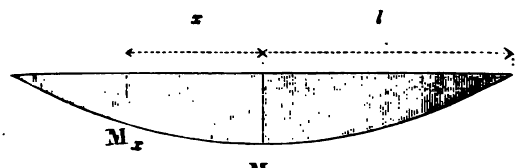
It now remains to find the stresses in the booms of the girders produced by the permanent load and the temperature load.

The part of the permanent load carried by the girders will evidently be uniformly distributed over the span. Therefore, if M_0 is the bending moment at the centre, the bending moment M_x at a distance x from the centre can be found from the equation (see equation 65):

$$M_x = M_0 \left(1 - \frac{x^2}{l^2}\right). \qquad [210]$$

This is the equation to a parabola, as shown in Fig. 486.

Fig. 486.



The stress S_p evidently also follows the same law as the corresponding bending moment M_x; and since the value of S_p at the centre is 1·685 kilo. per square millimetre (see equation 176),

$$S_p = 1\cdot685 \left(1 - \frac{x^2}{l^2}\right). \qquad [211]$$

Further, the stress S_t due to the temperature load, can be similarly obtained, since this load is also uniformly distributed over the span. And by equation 167 the value of this stress at the centre is 2·996 kilos., therefore,

$$S_t = 2\cdot996 \left(1 - \frac{x^2}{l^2}\right). \qquad [212]$$

The following table has been constructed from equations 211 and 212, and from the result obtained in § 58 :—

Values of $\frac{x}{l}$	Stresses in the Booms of Girders produced by			
	Moving Load.	Permanent Load.	Temperature Load.	Total Stress.
	S_m	S_p	S_t	$S_m + S_p + S_t$
0	0·95	1·685	2·996	5·63
0·0612	0·98	1·68	2·98	5·64
0·125	1·07	1·66	2·95	5·68
0·2	1·15	1·62	2·88	5·64
0·24	1·20	1·59	2·82	5·61
0·25	1·22	1·58	2·81	5·61
0·408	1·34	1·40	2·48	5·24
0·5	1·344	1·26	2·25	4·85
0·6	1·28	1·08	1·92	4·28
0·75	1·0	0·74	1·31	3·05
1	0	0	0	0

This table shows that the maximum stress occurs when $x = \cdot 125\, l$, and that it is then equal to 5·68 kilos. per square millimetre. It must, however, be observed that these stresses will be further increased by the pressure of the wind, and also by the extension of the land ties. The effect of both these causes will be treated of in § 62 and § 63.

§ 60.—Calculation of the Shearing Stress produced by the Moving Load.

The table at the end of § 57 shows that for every value of x there is only one zero-point. This zero-point forms a loading boundary; there is, however, a second loading boundary, which is situated at the section under consideration itself, for a load can produce no shearing force at the section immediately below it.

Thus, for instance, Fig. 487 shows the arrangement of the load that gives the greatest shearing force V_m at the section whose distance is $x = 0\cdot 75\, l$ to the left of the centre. Now, from equation 189,

$$V_m = D - n m x;$$

and it will be found by means of equation 184 that,

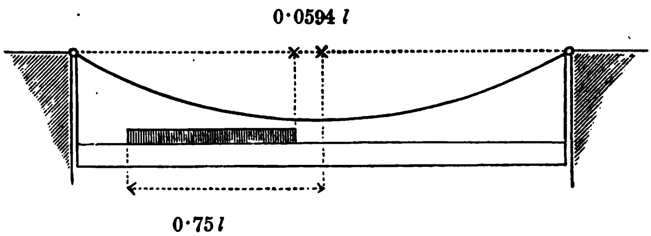
$$2n = 0 \cdot 7376.$$

Therefore (see equation 190):

$$V_m = 0 \cdot 2 \frac{(0 \cdot 75\, l - 0 \cdot 0594\, l)}{2} \left(1 + \frac{0 \cdot 75\, l + 0 \cdot 0594\, l}{2\, l}\right)$$

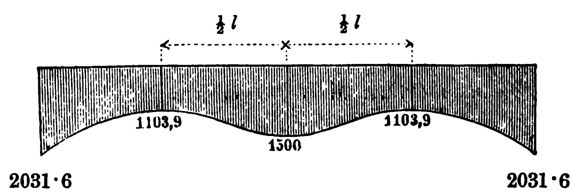
$$- 0 \cdot 3688 \times 0 \cdot 2 \times 0 \cdot 75 = 1250 \cdot 8 \text{ kilos.}$$

FIG. 487.



If the value of V_m, for a series of values of x, be calculated, and the results plotted, the curve shown in Fig. 448 will be obtained. This curve shows that the vertical shearing force V_m due to the moving load, has one maximum and one minimum

FIG. 488.



value on each side of the centre and a maximum value at the centre. The greater maximum (strictly speaking, it is not a maximum) occurs at the abutments, and is 2031·6 kilos.; the smaller maximum is at the centre, and its value is 1500 kilos. The minimum is at a distance $\frac{1}{2}\, l$ from the centre, and is equal to 1103·9 kilos.

§ 61.—Calculation of the Shearing Stresses due to the Permanent and Temperature Loads, and of the Maximum Stresses in the Braces.

The permanent load on the bridge produces a uniformly distributed load of 42·13 kilos. per metre run on the girders (see equation 174). The shearing force at any section of the bridge can therefore be found by means of equation 66A, and by substituting the various values it will be found that

$$V_p = 1263 \cdot 9 \frac{x}{l}. \qquad [213]$$

The temperature load being also uniformly distributed, the shearing stress V_t at any section can be found from the same equation. This load amounts to 75 kilos. per metre run. Hence:

$$V_t = 2250 \frac{x}{l}. \qquad [214]$$

The results obtained are embodied in the following table:—

Values of $\frac{x}{l}$	Vertical Shearing Force produced by			
	Moving Load.	Permanent Load.	Temperature Load.	Total.
	V_m	V_p	V_t	$V_m + V_p + V_t$
0	1500	0	0	1500
0·25	1419·5	316·0	562·5	2298·0
0·5	1103·9	631·9	1125·0	2860·8
0·75	1250·8	947·9	1687·5	3886·2
1·0	2031·6	1263·9	2250	5545·5

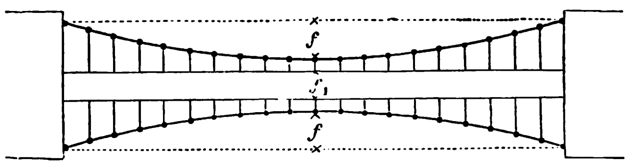
If the girder is divided into square bays by vertical braces, and if in each bay two diagonals are placed which can only resist tension, the verticals will be in compression. On referring to the above table, it will be seen that the compression in the vertical over the abutments in each girder is $\frac{5545 \cdot 5}{2} =$ 2772·75 kilos., and that the stress in the succeeding verticals gradually decreases as far as to the centre, where its value is

$\frac{1500}{2} = 750$ kilos. The tension in the diagonals is $\frac{1500 \times \sqrt{2}}{2}$ $= 1060 \cdot 7$ kilos. in those of the central bay, and it increases towards the abutments, where it is $\frac{5545 \cdot 5 \times \sqrt{2}}{2} = 3921 \cdot 3$ kilos.

§ 62.—Calculation of the Stresses in the Wind-Stays and Wind-Braces.

A parabolic form can be given with advantage to the wind-stays, connecting them by horizontal rods to the lower joints of the girder, as shown in Fig. 489. The lower booms are braced together, and thus a combination of a girder with a suspension chain, similar to the main structure, is obtained.

Fig. 489.



As before, the first step is to find the economical height of arc of the wind-stay, and this can be done by writing f instead of h, and $f_1 = 2 \cdot 25$ metres (the breadth of the bridge) instead of h_1 in equation 155, thus:

$$\frac{f}{2 \cdot 25} = \frac{\frac{3}{4}(\delta + \Delta)}{\frac{5}{6}\delta_1}; \qquad [215]$$

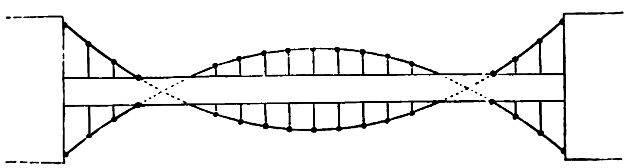
where δ is the safe extension of the wind-stays, Δ the extension due to temperature, and δ_1 the safe extension in the girder per unit of length; and these values are to be taken independently of the extensions produced by the vertical loads. The following table has been computed from the above equation by giving different values to δ_1, δ, and Δ.

δ_1	δ	Δ	f
			m.
$\frac{6}{20000}$	$\frac{6}{20000}$	$\frac{1}{4000}$	3·7125
$\frac{6}{20000}$	$\frac{6}{20000}$	$\frac{1}{2000}$	5·4
$\frac{6}{20000}$	$\frac{8}{20000}$	$\frac{1}{4000}$	4·3875
$\frac{6}{20000}$	$\frac{9}{20000}$	$\frac{1}{4000}$	4·725
$\frac{6}{20000}$	$\frac{9}{20000}$	$\frac{1}{2000}$	6·4125
$\frac{5}{20000}$	$\frac{10}{20000}$	$\frac{1}{4000}$	6·075
$\frac{5}{20000}$	$\frac{10}{20000}$	$\frac{1}{2000}$	8·1

In choosing a value for f from this table, it should be considered whether it is probable, or even possible, that all the unfavourable circumstances can occur simultaneously. If they do occur simultaneously, then one of the larger values of f must be taken.

Now, it is most unlikely that a high wind will be blowing when the temperature is at its maximum, and further, a rise of temperature of 20°·5 C. ($\Delta = \frac{1}{4000}$) above the mean is really an ample allowance. It would therefore appear that f can be made 4 metres, although it must be admitted that a greater height of arc would be preferable, if the breadth of the abutments will allow of it. It will be observed, however, that a greater height of arc can be obtained, as indicated in Fig. 490.

FIG. 490.



[NOTE.—It is thought that this arrangement of the wind-stays is open to the following objections. The bars connecting the centre joints of the lower booms to the chains, are struts, and would therefore require a larger scantling than the corresponding ties in the arrangement shown in Fig. 489. Further, these struts are in unstable equilibrium—that is, with the slightest displacement of the end attached to the chain a tendency to turn about the other end would arise. These struts would therefore have to be braced up to the side of the main girders.]

The distribution of the wind-pressure between the chain and the horizontal girder can be found from equation 171, by writing f instead of h, f_1 instead of h_1, ϕ the sectional area of the wind-stay instead of F, and $\frac{1}{2}$ F_1 instead of F_1 (the reason of this last alteration is that only the two bottom booms are connected to form the horizontal girder), thus:

$$n = \frac{1}{1 + \frac{9}{40} \cdot \frac{E_1}{E} \cdot \frac{F_1}{\phi} \cdot \frac{f_1^2}{f^2}}. \qquad [216]$$

If $E = E_1$, $F_1 = 15000$ sq. millimetres, $\phi = 1250$ sq. millimetres, $f = 4000$ millimetres, $f_1 = 2250$ millimetres, it will be found that

$$n = 0 \cdot 5393. \qquad [217]$$

Let it be assumed that the wind-pressure is $w = 0 \cdot 2$ kilo. per millimetre run; then the part carried by the wind-stay will be

$$0 \cdot 5393 \times 0 \cdot 2 = 0 \cdot 10786 \text{ kilo. per millimetre run}; \qquad [218]$$

and by the horizontal girder,

$$(1 - 0 \cdot 5393) \times 0 \cdot 2 = 0 \cdot 09214 \text{ kilo. per millimetre run}. \qquad [219]$$

The stress at the centre of the wind-stay can be found by equation 175, thus:

$$S = \frac{0 \cdot 10786\, l^2}{2\, \phi f} = 9 \cdot 7 \text{ kilos. per sq. millimetre}; \qquad [220]$$

and, from equation 176, the stress produced in the lower booms of the lattice girders is

$$S_1 = \frac{0 \cdot 09214\, l^2}{\frac{1}{2}\, F_1 f_1} = 4 \cdot 9 \text{ kilos. per sq. millimetre}. \qquad [221]$$

The stresses produced by temperature in the wind-stays and in the horizontal girder can be found in the manner explained in § 54. Thus, if the range of temperature on each side of the mean is 41° C., it will be found from equation 165 that

$$k = 0 \cdot 0512 \text{ kilo. per millimetre run}; \qquad [222]$$

that is to say, when the temperature is 41° C. above the mean, the girder will have a temperature load of 51·2 kilos. per metre run, and when the temperature is 41° C. below the mean the wind-stay will have a temperature load of the same amount (supposing that there is no temperature load at the mean temperature).

These loads produce the following stresses:
In the wind-stay,

$$S' = 4 \cdot 608 \text{ kilos. per sq. millimetre;} \qquad [223]$$

in the lower booms of the girders,

$$S'_1 = 2 \cdot 73 \text{ kilos. per sq. millimetre.} \qquad [224]$$

The maximum horizontal load on the girder, due to the wind and change of temperature conjointly, is

$$(1 - n)\, w + k = 0 \cdot 14334 \text{ kilo. per millimetre.} \qquad [225]$$

The maximum shearing stress at the abutments is therefore

$$[(1 - n)\, w + k]\, l = 4300 \text{ kilos.} \qquad [226]$$

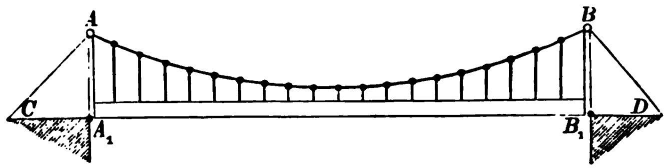
It will be seen from the table at page 369 that the maximum shearing stress at the abutments on each of the vertical girders is $\frac{5545 \cdot 5}{2} = 2773$ kilos. If, therefore, the braces in the horizontal girder are arranged in the same manner as those in the vertical girders, the stresses in them will be to those in the vertical girders as 4300 : 2773, and their sections will have to be made larger accordingly.

§ 63.—Influence of the Extension of the Back-stays.

In the previous calculations, the points of attachment of the chains A and B (Fig. 491) were considered as absolutely fixed. If, however, A and B are placed at the top of vertical columns $A A_1$ and $B B_1$ capable of free rotation about their lower points A_1 and B_1, back-stays A C and B D must be intro-

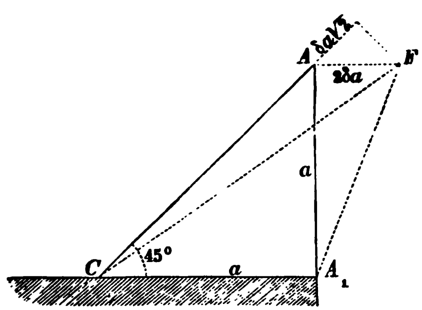
duced to keep these columns in position. The extension of these stays will allow the points A and B to move slightly towards the centre, and the lowest point of the chain A B will

FIG. 491.



be slightly lowered in consequence, and this will necessitate a correction in the distribution of the loads already found.

FIG. 492.



If the original angle of inclination of the back-stay A C (Fig. 492) is 45°, and its extension per unit of length δ, the horizontal displacement of the point A is evidently

$$A F = 2 \delta a;$$

and the deflection at the centre of the main chains due to this displacement is (similarly to equation 148),

$$s = \tfrac{3}{4} (2 \delta a) \frac{l}{h} \text{ (approximately).} \qquad [227]$$

Let D represent the extension per unit of length in the

main chains that would produce the same deflection (supposing A and B to be fixed points), then, according to equation 148,

$$s = \frac{8}{5} D \frac{l^2}{h}. \qquad [228]$$

Equating these two values of s,

$$\frac{D}{\delta} = 2 \frac{a}{l}; \qquad [229]$$

or substituting $a = 4$ metres and $l = 30$ metres,

$$\frac{D}{\delta} = \frac{4}{15}. \qquad [230]$$

Applying this result to equation 165, it will be seen that to find the temperature load transmitted from the main chains to the girders, when the extension of the back-stays per unit of length is the same as that of the main chains, it is only necessary to multiply the previous result by $1 + \frac{4}{15}$. (It will be observed that equation 230 is true for negative, as well as for positive values of δ.)

It was found, by means of equation 165, that an increase of temperature of 41° C. produced a temperature load of 75 kilos. per metre run on the girders. Therefore, owing to the simultaneous extension by temperature of the back-stays, this load will be increased by $\frac{4}{15} \times 75 = 20$ kilos. per metre run, and the total load will be 95 kilos. per metre run. The stress in the booms of the girders at the centre corresponding to the former temperature load was 3 kilos. per sq. millimetre; it will now be increased to

$$S'_t = (1 + \tfrac{4}{15})3 = 3\cdot 8 \text{ kilos. per sq. millimetre.} \qquad [231]$$

The former vertical shearing force at the abutments, due to temperature, was 2250 kilos. It now becomes

$$V'_t = (1 + \tfrac{4}{15})2250 = 2850 \text{ kilos.} \qquad [232]$$

It will be observed that there is an important difference between the extension in the back-stays, due to elasticity, and that due to temperature; for the maximum increase of load due

to the latter can be taken either positively or negatively, and can occur under any conditions of loading of the bridge; whereas the former can only be taken in the positive sense, and its maximum effect can only occur when the chain is fully loaded, and this happens precisely when the temperature is a minimum.

The effect of the extension of the back-stays on the distribution of the permanent and moving loads can be found as follows :—

The stress in the back-stays, when the angle of inclination is 45°, is $\sqrt{2}$ times greater than the horizontal stress in the main chains. If, however, their section be made $\sqrt{2}$ times greater than that of the main chains at the lowest point, the stress per unit of area will be the same, and therefore also the extension per unit of length.

In this case, as will be seen from equation 230, the extension of the back-stays has the same effect as if the extension of the main chains had been increased in the ratio of 1 to $1 + \frac{4}{15}$, or what amounts to the same, as if the modulus of elasticity of the main chains had been diminished in the ratio of $1 + \frac{4}{15}$ to 1. The distribution of the loads, when the extension of the back-stays is taken into account, can therefore evidently be found by substituting $\frac{15}{19}$ E for E in the various equations already obtained. Thus, from equation 171, the coefficient of load distribution for the permanent load becomes

$$n_1 = 0 \cdot 86184 \text{ (instead of } n = 0 \cdot 887656\text{)}. \qquad [233]$$

[NOTE.—This value of n_1 can also be obtained as follows :—

The deflection of the centre of the chains consists of two parts, one due to the extension of the back-stays and the other to the extension of the chains themselves, and their amount is given in equations 227 and 168 respectively.

Now, since it is assumed that the sectional area of the back-stays is $\sqrt{2}$ times that of the main chains, and consequently the value of δ is the same in both, δ in equation 227 can be replaced by its value $\frac{n_1 p l^2}{2\text{EF}h}$ obtained from equation 158, by writing $n_1\, p$ instead of k. Hence the total deflection at the centre of the main chains is

$$\frac{3}{4}\frac{n_1 p l^2 a}{\text{E F} h^2} + \frac{5}{8}\frac{n_1 p l^4}{\text{E F} h^3}.$$

This deflection must be equated to that at the centre of the girders given by equation 169, thus:

$$\tfrac{2}{3}\frac{n_1 p l^3 a}{\text{E F} h^2} + \tfrac{5}{8}\frac{n_1 p l^4}{\text{E F} h^2} = \tfrac{5}{8}\frac{(1 - n_1) p l^4}{\text{E}_1 \text{F}_1 h_1^2},$$

from which it will be found that

$$n_1 = \frac{1}{1 + \frac{8}{30} \cdot \frac{2a + l}{l} \cdot \frac{\text{E}_1 \text{F}_1 h_1^2}{\text{E F} h_2}}.$$

Now,

$$\frac{2a + l}{l} = \frac{2 \times 4 + 30}{30} = \tfrac{19}{15},$$

therefore,

$$n_1 = \frac{1}{1 + \frac{8}{30} \cdot \frac{\text{E}_1 \text{F}_1 h_1^2}{\frac{15}{19}\text{E F} h_2}} = 0 \cdot 86184.]$$

The part of the load taken by the girders will be therefore increased in the ratio,

$$\frac{1 - n_1}{1 - n} = \frac{0 \cdot 13816}{0 \cdot 11234} = 1 \cdot 23, \qquad [234]$$

or 23 per cent.; and the stresses in the booms and the shearing forces will be increased in the same proportion. The stress S_p in the centre of the booms was 1·685 kilos. per sq. millimetre; it now becomes

$$\text{S}'_p = 1 \cdot 685 \times 1 \cdot 23 = 2 \cdot 07 \text{ kilos. per sq. millimetre.} \qquad [235]$$

The vertical shearing force at the abutments was 1263·9 kilos.; it is now

$$\text{V}'_p = 1263 \cdot 9 \times 1 \cdot 23 = 1554 \cdot 3 \text{ kilos.} \qquad [236]$$

In applying this correction to the stresses produced by the moving load, it is to be observed that the most unfavourable arrangements of the load will be slightly altered; or, in other words, that the zero-points will be shifted. It will be found, for instance, that instead of $u = 0 \cdot 5715\, l$ in Fig. 481, $u = 0 \cdot 685\, l$ gives the position of the load when the stress S_m in the booms at the centre of the girders is a maximum, as will

appear from equations 196 and 198, in which E must be replaced by $\frac{16}{9}$ E. As before, it is found that in this case

$$n = 0 \cdot 7609,$$

and

$$S'_m = 1 \cdot 12 \text{ kilos. per sq. millimetre (instead of } 0 \cdot 95 \text{ kilos).} \qquad [237]$$

Similarly, when $x = \frac{1}{2} l$, the corrected value of n is $0 \cdot 4082$, and $S'_m = 1 \cdot 445$.

The maximum shearing force V_m was $2031 \cdot 6$ kilos., to which corresponded $v = 0 \cdot 2616\, l$ (see tables, pages 362 and 369). The corrected value is $v = -\ 0 \cdot 245\, l$, and

$$V'_m = 2373 \text{ kilos.} \qquad [238]$$

§ 64.—Recapitulation of the Results of the Calculations.

In the preceding calculations the following dimensions, &c., were assumed or found :—

Permanent load	$p = 375$ kilos. per metre run.
Moving "	$m = 200$ " " "
Wind-pressure	$w = 200$ " " "
Sum of the sectional areas of the main chains	$F = 7500$ sq. millimetres.
Sum of the sectional areas of the back-stays	$F \sqrt{2} = 10600$ sq. millimetres.
Sectional area of each of the wind-stays	$\phi = 1250$ sq. millimetres.
Sum of the sectional areas of the four booms of the girders	$F_1 = 15000$ " "
Height of arc of the main chains ..	$h = 4$ metres.
Depth of the girders	$h_1 = 1 \cdot 5$ metres.
Height of arc of the wind-stays ..	$f = 4$ metres.
Width of the bridge	$f_1 = 2 \cdot 25$ metres.
Span of the bridge	$2\, l = 60$ metres.

The tension in the lower boom of either of the girders will be greatest when the moving load is in its most unfavourable position, the temperature highest, and the wind-pressure greatest (the wind blowing against the other girder). From equations

235, 237, 231, 219, and 224, it will be found that the stresses due to these various causes are:

$S'_p = 2 \cdot 07$ kilos. per sq. millimetre .. .. .. (permanent load).
$S'_m = 1 \cdot 12$ kilos. „ „ .. .. .. (moving load).
$S'_t = 3 \cdot 8$ kilos. „ „ .. .. .. (temperature load).
$S'_w = 4 \cdot 9 + 2 \cdot 73$ „ „ (wind-pressure occurring with highest temperature).

Adding these together, the greatest tension in the lower booms is found to be

$$S_{(max.)} = 14 \cdot 62 \text{ kilos. per sq. millimetre.} \qquad [239]$$

The compression in the upper booms will also reach its greatest value under these circumstances, and can be found by adding together the first three stresses, amounting to 7 kilos. per sq. millimetre (the wind-pressure has no effect on the upper booms).

The maximum vertical shearing force at the abutments is found by adding together the values given in equations 236, 238, 232, thus:

$V'_p = 1554$ kilos. .. .. (permanent load).
$V'_m = 2373$ kilos. .. .. (moving load).
$V'_t = 2850$ kilos. .. .. (temperature load).

or

$$V_{(max.)} = 6777 \text{ kilos.} \qquad [240]$$

And this is evidently also the maximum stress in the bars $A A_1$ or $B B_1$ (Fig. 463) to which the ends of the girders are attached.

These bars, it was seen, have also to act as struts; it is therefore necessary to find the minimum stress in them. Now the vertical shearing force produced by the moving load alone when covering the whole bridge, is evidently $\frac{200}{375} \times 1554 =$ 829 kilos. Thus the minimum added to the maximum shearing force produced by the moving load will be equal to 829 kilos.

Therefore,

$$V'_m = 829 - 2373 = -1544 \text{ kilos.}$$

The minimum stress in the bars $A\,A_1$ or $B\,B_1$ is therefore

$$V_{(min.)} = +1554 - 1544 - 2850 = -2840 \text{ kilos.}; \qquad [241]$$

and they must consequently be strong enough to resist a thrust of 2840 kilos.

The stress in the main chains reaches its maximum value when the bridge is fully loaded, and the temperature is lowest. The total load on the bridge is 575 kilos. per metre run, and of this the chains carry, according to equation 233, $0 \cdot 862 \times 575 = 495 \cdot 56$ kilos. per metre. The temperature load is 95 kilos. per metre (see page 375). In the most unfavourable case, therefore, the chains have to carry $590 \cdot 56$ kilos. per metre of the horizontal projection, and the corresponding stress at the lowest points of the chains is, from equation 166,

$$S = \frac{0 \cdot 59056 \times 30000^2}{2 \times 4000 \times 7500} = 8 \cdot 86 \text{ kilos. per sq. millimetre.} \qquad [242]$$

If the section of the chains is constant, the stress at the abutments will be (see § 8)

$$8 \cdot 86 \sqrt{1 + (\tfrac{8}{30})^2} = 9 \cdot 13 \text{ kilos. per sq. millimetre.} \qquad [243]$$

The stress in the back-stays will, however, evidently be $8 \cdot 86$ kilos. per sq. millimetre, since their section is $\sqrt{2}$ times that of the main chains.

If it be assumed that the weight of the chains is 4500 kilos., then the sum of the stresses on all the verticals connecting the chains and the girders is

$$\overset{\text{(max. load on the chains)}}{590 \cdot 56 \times 60} - 4500 = 30934 \text{ kilos.};$$

and the stress in each will therefore be $\dfrac{30934}{N}$, where N is their number.

The wind-stays are under the worst conditions when the

wind-pressure is greatest and the temperature is lowest. Hence from equations 218 and 223 the maximum stress in the centre of the wind-stays is,

$$9 \cdot 7 + 4 \cdot 6 = 14 \cdot 3 \text{ kilos. per sq. millimetre.}$$

From equation 226 it will be seen that the greatest shearing force taken up by the horizontal wind-braces, is 4300 kilos.

And lastly, the maximum tension in each of the horizontal rods connecting the wind-stays with the girders, is

$$\frac{(107 \cdot 86 + 51 \cdot 2) \times 60}{N_1} = \frac{9540}{N_1},$$

according to equations 218 and 222, N_1 being their number.

§ 65.—Adjustment of the Vertical Rods connecting the Girders with the Suspension Chains.

When investigating the effect of the permanent load, it was assumed that the structure was weightless and the girders perfectly straight. The permanent load was then applied, and the centre of the bridge consequently deflected by a certain amount. Now, on account of the compound nature of the structure, it is only by chance that this will be the actual deflection of the structure when subject to its own weight, and it is only in this case that the distribution of the load found by means of equation 171, will be the true one. In fact, it is evident that by shortening the vertical rods connecting the girders with the suspension chains, the part of the load carried by the former will be diminished, and the reverse effect will be obtained by lengthening these rods.

It therefore becomes a question whether by altering the length of these rods the stresses cannot be more uniformly distributed between the various parts of the structure.

The deflection at the centre of the girders produced by the permanent load alone, as found from equation 169, is (sub-

stituting however n_1 for n (from equation 233) to allow for the extension of the back-stays),

$$s = \tfrac{5}{8}\,\frac{0\cdot13816 \times 0\cdot375 \times 30000^4}{20000 \times 15000 \times 1500^2} = 51\cdot81 \text{ millimetres}; \qquad [244]$$

and from equation 235 the stress in the booms corresponding to this deflection is $S'_p = 2\cdot07$ kilos. per sq. millimetre. But as already explained, this will only be the actual stress in the girders if, when put up, the deflection is 51·81 millimetres, supposing that the girders are perfectly straight when in the condition of no stress. But if, after erection, the girders are made straight again by shortening the vertical rods by means of set-screws, the stress $S'_p = 2\cdot07$ kilos. per sq. millimetre will disappear, and the maximum stress (equation 239) will be reduced from 14·62 to 12·55 kilos. per sq. millimetre. At the same time the part of the permanent load, viz. 51·81 kilos. per metre, formerly carried by the girders, will be supported by the suspension chains.

The tightening of the screws may however be continued until, for instance, an upward deflection of 51·81 millimetres has been obtained; the maximum stress in the booms would thereby be further reduced by 2·07 kilos. and would become 10·48 kilos. per sq. millimetre, and the total increase of load on the chains would then be $2 \times 51\cdot81 = 103\cdot62$ kilos. per metre of the horizontal projection.

Tightening up the set-screws, although it diminishes the stress due to a positive bending moment, evidently increases by the same amount the stress due to a negative bending moment; that is, when the girder is bent upwards.

The limit to which the set-screws may be tightened up with advantage is therefore reached, when the greatest positive bending moment is equal to the greatest negative bending moment.

The greatest negative bending moment under the original circumstances occurs at the centre of the bridge, for although, according to the table at page 367, the moving load produces its maximum effect at a distance $x = \frac{1}{2}\,l$ from the centre, yet it

will be found that the other causes are sufficient to make the negative bending moment greatest at the centre.

If S is the stress in the lower booms at the centre produced by the moving load alone when covering the whole bridge, it is evident that $\frac{S}{S'_p} = \frac{200}{375}$. But $S_p = 2\cdot07$ kilos. per sq. millimetre. Hence

$$S = \tfrac{200}{375} \times 2\cdot07 = 1\cdot105 \text{ kilos.} \qquad [245]$$

Now the sum of the maximum and minimum stresses produced by the moving load must be equal to S. Hence

$$S_{(min.)} = S'_m - S = 1\cdot12 - 1\cdot105 = -0\cdot015 \text{ kilos. per sq. millimetre} \quad [246]$$

To this negative stress must now be added the negative stress due to the lowest temperature, $-3\cdot8$ kilos. (equation 231), the negative stress produced by the wind-pressure at the lowest temperature, $-(4\cdot9 - 2\cdot73) = -2\cdot17$ kilos. (equations 221 and 224), and lastly, the positive stress, $+2\cdot07$ kilos. produced by the dead load, thus:

$$\begin{aligned} S_{(min.)} &= -0\cdot015 - 3\cdot8 - 2\cdot17 + 2\cdot07 \\ &= -3\cdot915 \text{ kilos. per sq. millimetre.} \end{aligned} \qquad [247]$$

Therefore, if the set-screws be tightened up until the girder is straight $S_{(min.)}$ will be increased to

$$-3\cdot915 - 2\cdot07 = 5\cdot985 \text{ kilos. per sq. millimetre;}$$

and if the tightening be further continued until the upward deflection of the girder is $51\cdot81$ millimetres $S_{(min.)}$ will become

$$-5\cdot985 - 2\cdot07 = -8\cdot055 \text{ kilos. per sq. millimetre.}$$

It thus appears that the set-screws may be tightened up with advantage until the centre of the girder deflects upwards $51\cdot81$ millimetres. For under the original conditions the stress at the centre in the lower booms varied from $+14\cdot62$ kilos. to $-3\cdot915$ kilos.; whereas now these limits will be $+10\cdot48$ kilos. and $-8\cdot055$ kilos.; and further, the stresses in the suspension chains will only be increased to $10\cdot41$ kilos. per square

millimetre at the centre, and to 10·73 kilos. at the points of attachment, representing a load of $590 \cdot 56 + 103 \cdot 62 = 694 \cdot 18$ kilos. per metre of horizontal projection.

[NOTE.—The ratio of the greatest stresses in the booms of the girders is:

$$\frac{10 \cdot 48}{8 \cdot 055} = \frac{5 \cdot 24}{4};$$

and since the ratio of the resistance of wrought iron to tension to its resistance to compression is approximately the same, it appears that the set-screws should not be tightened any further.]

The total maximum load on the rods connecting the chains and the girders will now be increased to

$$30934 + 60 \times 103 \cdot 62 = 37151 \text{ kilos.};$$

and the stress in each will therefore be $\dfrac{37151}{N}$.

The vertical shearing force at the end of the girders will be diminished by $30 \times 103 \cdot 62 = 3109$ kilos., and therefore (equations 240 and 241):

$$V_{(max.)} = + 6777 - 3109 = 3668 \text{ kilos.} \qquad [248]$$
$$V_{(min.)} = - 2840 - 3109 = - 5949 \text{ kilos.} \qquad [249]$$

Therefore under the new conditions, the bars $A\,A_1$ or $B\,B_1$ in Fig. 463 must be capable of bearing a thrust of 5949 kilos.

Since the deflection of 51·81 millimetres at the centre of the girders corresponds to a load on the girders of 51·81 kilos. per metre, or of 375 kilos. per metre on the whole bridge, it follows that the upward deflection of 51·81 millimetres will disappear when a load of 375 kilos. per metre has been applied to the bridge. Therefore, to ensure the above distribution of the stresses the following can be specified: "The deflection at the centre of the girders is to be zero when a load of 375 kilos. per metre run is placed on the bridge at the mean temperature."

A temperature load on the girders of 51·81 kilos. will evidently produce the same effect. Now an increase of temperature of 41° C. loads the girders with 95 kilos. per metre;

therefore, the temperature at which the temperature load will be 51·81 kilos. per metre, is :

$$t = 41 \times \frac{51 \cdot 81}{95} = 21^\circ \cdot 24 \text{ C.} \qquad [250]$$

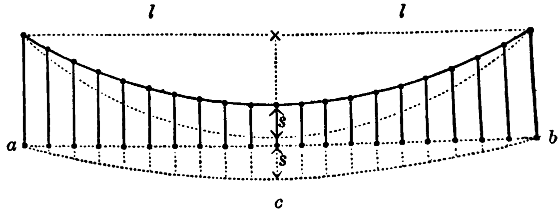
The desired result can therefore be obtained if, when the temperature is 21°·24 C., the girders are made straight by means of the set-screws.

§ 66.—Remarks on the Degree of Accuracy of the Method employed.

The method adopted to calculate the coefficient of load-distribution n is only approximate. But, as will be seen, the errors involved are very small, and to a certain extent they balance each other; they are therefore of no practical importance. To prove this, it will be sufficient to consider the simpler case given in Fig. 463, in which the points of attachment of the chains are considered fixed, and the difference between the value of n found from equation 172, and its corrected value, can be considered as a measure of the error.

It will be observed that equation 170, obtained by equating the deflection at the centre both of the girders and the chains, is not strictly accurate. For if the bottom ends of the rods connecting the girders with the chains were free, they would,

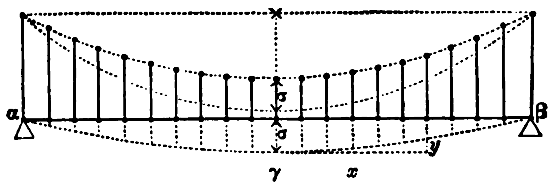
Fig. 493.



when the chains deflected, be in a parabola (Fig. 493), whereas their points of attachment to the girders would be in an elastic curve (Fig. 494). These two curves cannot cover each other,

and the actual deflection curve of the girders will lie between them. It would therefore appear more accurate to equate the mean deflections instead of the ordinates σ and s at the centre.

FIG. 494.



The mean deflection will be the mean ordinate of the curve, and this mean ordinate can be defined to be the height of the rectangle on the same base and of the same area as the area enclosed between the curve and the axis of x. In the present case, since the base of both curves is the same, $= 2\,l$, the result can be attained by equating the areas themselves.

The area of the part of the parabola $a\,b\,c$ in Fig. 493 is:

$$J = \tfrac{2}{3}\, 2 l s; \qquad [251]$$

and by substituting the value of s from equation 168:

$$J = \frac{n\,p\,l^4}{2\,E\,F\,h^2}. \qquad [252]$$

Again, the height of arc of the elastic curve $a\,\gamma\,\beta$ (Fig. 494) is, according to equation 169:

$$\sigma = \tfrac{5}{6}\,\frac{(1-n)\,p\,l^4}{E_1\,F_1\,h_1^{\,2}}. \qquad [253]$$

And the equation to this curve is: *

$$y = \frac{(1-n)\,p}{E_1\,F_1\,h_1^{\,2}}\left(l^2\,x_2 - \frac{x^4}{6}\right). \qquad [254]$$

But the area comprised between the curve $a\,\beta\,\gamma$ and the axis of x is:

$$J - 2\int_0^l (\sigma - y)\,d\,x. \qquad [255]$$

* This equation can be obtained by a process similar to that employed at p. 329.—TRANS.

Hence, substituting for σ and y, and integrating between the limits:

$$J = \tfrac{16}{15} \frac{(1-n)\, p\, l^5}{E_1 F_1 h_1^2}.$$

Equating both values of J:

$$n = \frac{1}{1 + \tfrac{15}{32} \cdot \frac{E_1}{E} \cdot \frac{F_1}{F} \cdot \frac{h_1^2}{h^2}}. \qquad [256]$$

Whence $n = 0 \cdot 88352$ (instead of $0 \cdot 887656$) and $1 - n = 0 \cdot 11648$ (instead of $0 \cdot 112344$). This correction therefore diminishes n, but even for $1 - n$ the error is only $3 \cdot 7$ per cent.

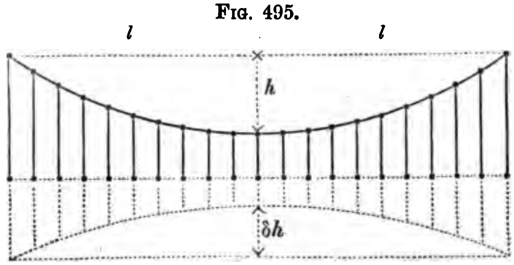
Secondly, the extension of the vertical rods connecting the chains with the girders was not taken into account. If the elongation of these rods is δ per unit of length, their lower extremities will lie in a parabola whose height of arc is δh (supposing that the suspension chains do not alter). Since this parabola has its convex side upwards, δh must be subtracted from the deflection s produced by the lengthening of the chains, and if, further, δ is also the extension per unit of length of the chains, the actual deflection at the centre of the span is

$$s - \delta h = \tfrac{3}{4} \delta \frac{l^2}{h} - \delta h; \qquad [257]$$

or more accurately, replacing the member $\frac{2}{3} \frac{h^2}{l^2}$ previously omitted (see equation 147):

$$s - \delta h = \tfrac{3}{4} \delta \frac{l^2}{h} \left(1 + \tfrac{2}{3} \frac{h^2}{l^2}\right) - \delta h. \qquad [258]$$

FIG. 495.



Substituting

$$\delta = \frac{H}{F E} = \frac{n p l^2}{2 E F h}$$

and reducing,

$$s - \delta h = \frac{8}{3} \frac{n p l^4}{E F h^2} \left(1 - \frac{2}{3} \frac{h^2}{l^2}\right) \qquad [259]$$

This value of $s - \delta h$ must be written instead of s in equation 168; it will then be found, by means of equation 170, that

$$n = \frac{1}{1 + \frac{8}{15} \cdot \frac{E_1}{E} \cdot \frac{F_1}{F} \cdot \frac{h_1^2}{h^2} \cdot \left(1 - \frac{2}{3} \frac{h^2}{l^2}\right)}. \qquad [260]$$

From this equation $n = 0 \cdot 88884$ (instead of $0 \cdot 887656$). Thus the second error partially neutralizes the first.

If both corrections be made, the more accurate formula for finding n is;

$$n = \frac{1}{1 + \frac{15}{32} \cdot \frac{E_1}{E} \cdot \frac{F_1}{F} \cdot \frac{h_1^2}{h^2} \cdot \left(1 - \frac{2}{3} \frac{h^2}{l^2}\right)},$$

whence $n = 0 \cdot 88474$ (instead of $0 \cdot 887656$), showing that the former value of n was only $\frac{1}{3}$ per cent. in error, and that the value of $1 - n$ is $2 \cdot 6$ per cent. in error. Obviously, therefore, the simpler method possesses ample accuracy for practical purposes.

APPENDIX.

a. Loads on Roofs and the Reactions at the Abutments caused by the Wind-pressure.

The manner of estimating the loads on roofs followed by Professor Ritter does not accord with the best and more recent English practice. Professor Ritter assumes that all the loads on a roof are vertical and evenly distributed over the surface. Now this is obviously erroneous as regards the wind-pressure, for it cannot act vertically on a roof nor on both sides at the same time. This manner of estimating the loads on roofs was, however, adopted by Tredgold, but in his time little was known about the pressure of wind.

Although the scantlings obtained for wooden roofs by means of Tredgold's assumption, coupled with a large factor of safety, have been proved by experience to be sufficiently strong, it cannot be inferred that this would be the case for iron roofs, at any rate, not for those of large span. And further, in iron roofs one end should be left free to move, to allow for the expansion and contraction produced by changes of temperature, a circumstance which affects the stresses due to the wind-pressure. A proper distribution of material is also of greater importance in an iron than in a wooden roof.

It is therefore necessary to arrive at some more accurate estimate of the loads to be borne by roofs.

These loads consist of:—

Permanent load, such as the weight of roof-covering, framework, and in some cases of the weight of a ceiling, lantern, &c.;

Variable load, the wind-pressure, and in some countries the weight of snow.

The weight of roof-covering, purlins, ceiling, &c., can always be readily obtained.* The only permanent load which is difficult to ascertain is that due to the weight of the truss itself. It can be found approximately if the weight of some similar structure is known. Or else approximate calculations can be made considering the roof truss to have no weight, and the scantlings thus obtained will give the required weight near enough for practical purposes, a correction then being made to the scantlings to allow for the weight of the roof truss.

The allowance to be made for the weight of snow will depend entirely on the locality in which the roof is to be erected. In this country it is not likely that snow will attain a greater depth than 6 in. on a roof when a strong wind is blowing, and this depth will also diminish as the pitch increases. An allowance of 5 lbs. per sq. ft. of horizontal surface covered would therefore seem ample,

---

* See Hurst's 'Architectural Surveyor's Handbook,' or Molesworth's 'Pocket Book of Engineering Formulæ.'

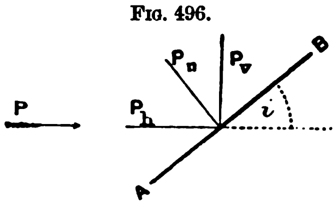
and it may also be assumed that the snow is uniformly distributed over the roof.

The following theory of the pressure of wind on roofs is due to Professor Unwin, and readers desirous of further information on the subject are referred to his works.

According to the mathematical definition, a perfect fluid can exert but a normal pressure on any body immersed within it, whether the body be at rest or in motion relatively to the fluid. Air, as is proved by experiment, is almost a perfect fluid, for, but a very slight tangential action is exerted on any body in motion within it. This tangential action is so insignificant that it need not be taken into account in the present case. It can therefore be assumed that the wind-pressure acts normally to the slope of the roof.

Let A B (Fig. 496) represent a plane surface perpendicular to the plane of the paper, and upon which air is impinging in a direction making an angle i with A B. Let P_n be the normal pressure per unit of area on the surface; this force can be resolved into its components, P_v and P_h, and evidently

FIG. 496.



$$P_n \sin i = P_h.$$

Now, Hutton by experiments made with his whirling machine measured the force P_h for different values of i, and found that

$$P_h = P \sin i^{\,1\cdot84 \cos i},$$

where P is the normal pressure per unit of area on a plane placed at right angles to the direction of motion of the air. Hence

$$P_n = \operatorname{cosec} i \; P \sin i^{\,1\cdot84 \cos i},$$

$$= P \sin i^{\,1\cdot84 \cos i - 1}. \qquad [a]$$

To apply this to the case of a roof, it remains to be determined what values should be given to i and P. The following data will be of use.

On one occasion during five years, the greatest pressure recorded at the Royal Observatory, Greenwich, reached 41 lbs. per sq. ft. At Bidston, near Birkenhead, a very exposed situation, the wind blew for one hour at the rate of 92 miles per hour, equivalent to 42¼ lbs. per sq. ft., and momentarily the pressure rose to 80 lbs. per sq. ft.

Although these pressures were recorded on anemometers placed in very exposed situations, and though it is probable that such wind-pressures are never reached in ordinary situations, yet, until this is actually proved, it would be unwise to make any reduction. Further, it is certain that the wind does not always blow horizontally, but since neither the limits of deviation are known, nor whether the intensity of pressure is changed or not when the direction of the wind is thus altered, it is probably best to assume that the wind blows horizontally, at the same time making an allowance by slightly increasing the value of P. It thus appears reasonable to assume P = 50 lbs.* per sq. ft., for it must be remembered that the pressure of 80 lbs. per sq. ft., registered at Bidston,

* Colonel Wray, R.E., in his 'Instruction in Construction,' assumes P = 50 lbs. per sq. ft., but Professor Unwin takes P = 40 lbs. per sq. ft.

occurred but momentarily and in a very exposed situation. To meet this greater wind-pressure, it may, however, be advisable to increase the scantling of those bars which are most affected by it. Since the wind has been assumed to blow horizontally, i will be the angle made by the slope of the roof with the horizontal, or, in other words, the pitch of the roof.

The wind-pressure can only act on one side of the roof at one time, and, owing to want of information on the subject, is assumed to be uniformly distributed (except, of course, in curved roofs). This assumption is, however, not altogether unfounded, for although, no doubt, eddies are produced by the walls of the building, &c., yet it is well known that a cushion of air is formed against the side of the roof which tends to equalize the pressure over the surface.

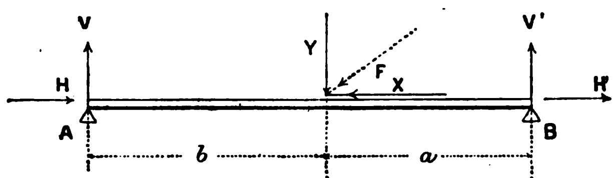
The following table * will be found of use in calculating the wind-pressure on roofs and the stresses caused thereby:—

i (pitch of roof).	P_n	P_v †	P_h †
°	lbs. per sq. ft.	lbs. per sq. ft.	lbs. per sq. ft.
5	6·3	6·1	0·5
10	12·1	12·0	2·1
20	22·6	21·3	7·8
30	33·0	28·5	16·5
40	41·6	31·9	26·8
50	47·6	30·6	36·5
60	50·0	25·0	42·5
70	51·3	17·5	48·1
80	50·5	8·8	49·8
90	50·0	0	50·0

The inaccuracy in the values of P_n for 60°, 70°, and 80°, is due to the empirical equation a being only approximate.

To resist the wind-pressure the supports of the roof must supply horizontal as well as vertical reactions, and these have to be determined before the stresses produced by the wind can be found.

Fig. 497.



Let Fig. 497 represent a body supported on two points in a horizontal straight line, and acted upon by a force F inclined to the vertical. This force can be replaced by its vertical and horizontal components X and Y, and the reactions at

* This table has been taken from 'Instruction in Construction,' by Colonel Wray, R.E.

† It is shown in a pamphlet by Professor Unwin, 'On the Effect of Wind-Pressure on Roofs,' that these values of P_v and P_h agree very well with some experiments made by Froude and Wenham.

the two points can also be replaced by their components V, H and V′, H′. The three conditions of equilibrium are:

$$
\begin{aligned}
X - H - H' &= 0, &&\text{resolving horizontally.}\\
Y - V - V' &= 0, &&\text{resolving vertically.}\\
V(a+b) - Y a &= 0, &&\text{moments round B.}
\end{aligned}
$$

Whence

$$H + H' = X.$$

$$V = \frac{a}{a+b} Y.$$

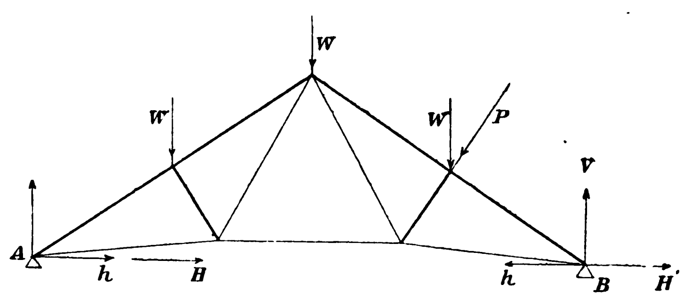
$$V' = \frac{b}{a+b} Y.$$

The vertical components V and V′ are therefore determinate, but the horizontal components H and H′ are indeterminate, the only condition necessary for equilibrium being that their sum is equal to the horizontal component X of F. This will be easily understood when it is considered that if the support A were smooth (mathematically), the whole of X would have to be supplied by B, and *vice versâ*. The values of H and H′ depend therefore on the nature of the supports.

Now, in iron roofs it is necessary to allow for the expansion and contraction arising from changes of temperature, and to do this, it is usual to fix one end of the truss, leaving the other free to move. In small roofs it is sufficient if the shoe at the free end simply rests on the template, but in large roofs a special arrangement of friction-rollers is generally provided.

Evidently the horizontal component of the reaction at the free end of the roof can never exceed the resistance to motion of that end. This resistance is μ V where μ is the coefficient of friction, and V the vertical reaction at the free end. In small roofs where no rollers are provided, μ will usually be the coefficient of friction of iron (cast) against stone, and may be taken at from ·4 to ·6. It may happen that, owing to a change of temperature, the free end of the roof is just on the point of motion. The full frictional resistance would thereby be called into play, and possibly the wind-pressure might produce a horizontal component equal and opposite to this resistance. For instance, let Fig. 498 represent a roof of which the end A is fixed and the end B is free to move. When the

FIG. 498.



temperature increases, horizontal forces h are generated at A and B, and by the above assumption, when the end B is on the point of motion $h = \mu V = H'$. Evidently, therefore, the horizontal reaction at A is

$$H + h = H + H' = X,$$

and at B it is zero. A similar case may occur when the wind is blowing from the left and the temperature diminishing. It is possible that this distribution of the horizontal reactions may produce greater stresses in some of the bars than the more normal distribution, and should therefore be considered. Thus in small iron roofs there are four cases to consider, namely,—

1. Wind on right. } Horizontal reaction at free end equal to friction.
2. Wind on left. }
3. Wind on right. } Horizontal reaction at free end zero.
4. Wind on left. }

In large roofs with a roller end, μ is so small (being the coefficient of rolling friction) that it may be neglected in the present inquiry. It may therefore be assumed that the horizontal component at the free end is always zero, and only two cases need be considered, namely,—

1. Wind on right, } Horizontal reaction at free end zero.
2. Wind on left, }

This will also meet the case of a roof truss having one end (the free end) supported by a column, and the other by a wall.*

In calculating the stresses in a roof it is best first to find the stresses produced by the weight of the roof-covering and framing when those occasioned by snow can generally be found by simple proportion. The stresses due to the wind are then to be ascertained as indicated above, and may be found by the "Method of Moments." If the results thus obtained are collected in a tabular form, the greatest tension or compression in each bar is easily found by inspection.

b. STABILITY OF PIERS AS REGARDS OVERTURNING.

In the calculations made both at p. 149 and p. 185 to ascertain the stability of the piers as regards overturning, Professor Ritter takes moments about the lower edge F of the pier. Since the resultant compression on the bed-joint at F does not appear in these equations of moments, it must act at F, or in other words, the total compression on the bed-joint is concentrated on the outer edge F. The intensity of pressure on this edge would therefore be so great that the material of which the pier is composed would be crushed. (Mathematically speaking, the edge is a line, and the pressure would therefore be infinite.) It is evident therefore that moments should not be taken round the outer edge of the pier, but about some axis inside the pier represented by the point E, Fig. 502, the position of this axis being such that the greatest intensity of pressure shall not exceed the safe resistance to crushing of the material (or of the mortar). It is proposed to find the position of E when the pier is rectangular on plan.†

In large structures of this kind the tenacity of the mortar should not be taken into account, for unequal settlements are liable to occur, which dislocate the joints. The pier will therefore be regarded as built with "uncemented blocks."

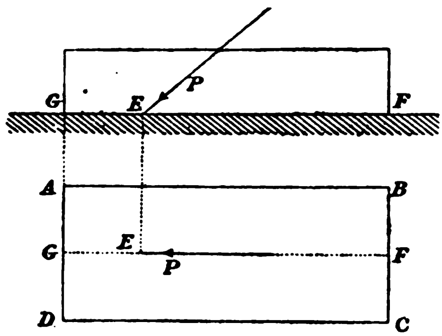
Now consider a body rectangular on plan (Fig. 499) pressed against a plane

* See last paragraph p. 35 of 'Lectures on the Elements of Applied Mechanics,' by Professor M. W. Crofton, F.R.S., in which this is pointed out.

† For further information on this subject see 'Applied Mechanics' and 'Civil Engineering,' by Professor Rankine; 'Engineering and Architecture,' by Rev. Canon Moseley; 'Instruction in Construction,' by Colonel Wray, R.E.

surface by a force P, the side of the body in contact being also plane. For simplicity, let the direction of P be as indicated in the figure, so that E the centre of pressure is situated on G F which is parallel to and equidistant from A B and D C. The pressure evidently cannot be uniformly distributed, unless E

FIG. 499.



bisects G F, but a fair assumption to make is that it varies uniformly. Or in other words, the pressure will reach its greatest intensity along the edge A D, and it will diminish uniformly towards the edge B C. And further, since E is equidistant from A B and D C, the intensity of pressure at all points on any line parallel to A D will be the same, and hence the pressure along G F represents that over the whole area.

Thus, if the position of E be such that the pressure at F is nothing, the triangle G F K (Fig. 500) will represent the intensity of normal pressure at every point in the line G F. The resultant normal pressure will evidently pass through the centre of gravity of this triangle, hence

$$\mathrm{GE} = \tfrac{1}{3}\,\mathrm{GF}.$$

FIG. 500.

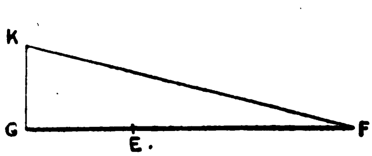
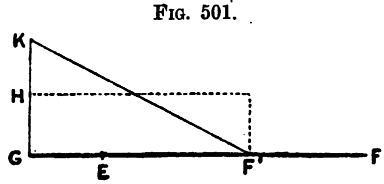


FIG. 501.



If however, $\mathrm{GE} < \tfrac{1}{3}\,\mathrm{GF}$ (Fig. 501), the pressure will vanish at a point F′, such that

$$\mathrm{GE} = \tfrac{1}{3}\,\mathrm{GF}',$$

and from F′ to F there will be no pressure.

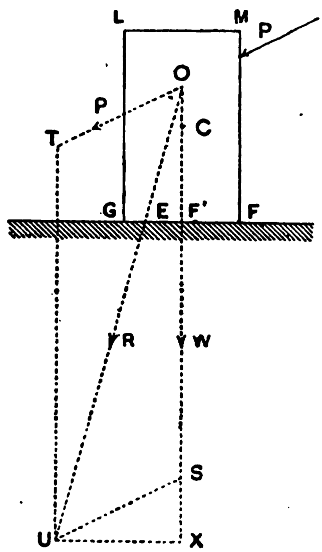
Evidently the maximum intensity of pressure is twice what it would be were the pressure uniformly distributed over G F′, for if H bisects G K the area of the *rectangle* H F′ is equal to that of the triangle G F′ K.

The position of E to fulfil the condition that the maximum intensity of

pressure K G should not exceed the *safe* resistance to crushing of the material, can probably be easiest ascertained by the graphic method.

For instance, let L M G F (Fig. 502) represent a pier rectangular on plan subject to a thrust P. The weight of the pier is W, and acts in the vertical through the centre of gravity C of the pier; O T and O S represent P and W in magnitude and direction, then by completing the parallelogram of forces the resultant R is obtained, represented by O U. The resolved part O X of R at right angles to G F is the total normal pressure on the bed-joint and the intersection E of O U and G F is the centre of pressure. Thus all the elements for finding the maximum intensity of pressure are known.

Fig. 502.



As a numerical example let the dimensions of the pier be: the breadth $b = 20$ feet, and the thickness G F = 6 feet. Let also W = 52 tons, P = 18 tons, the angle T O C = 24°, and the height of O above G F = 7 feet. It is then found by measurement that O X = 60 tons and G E = 1 foot, or G F′ = 3 feet as previously explained. Hence the greatest intensity of stress along the edge of the pier represented by G is

$$= 2 \cdot \frac{OX}{b \cdot GF'}$$
$$= 2 \cdot \frac{60}{20 \cdot 3}$$
$$= 2 \text{ tons per square foot.}$$

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